CS719 Graded Homework #1

Max Marks: 40 **Due date: Aug 13, 2010 (5 pm)**

- *Be brief, complete and stick to what has been asked.*
- *If needed, you may cite results/proofs covered in class without reproducing them.*
- ***Discussion among students is fine, but the solution you turn in must be your own solution in your own words. Cases of copying or indulgence in unfair means will be severely penalized, including award of FR grade.***

Problems on Proposition and First order logics:

1. We have learnt about clauses and conjunctive normal forms (CNF) of propositional logic formulae in class. A clause is said to be a *Horn clause* if it contains at most one positive (or unnegated) literal. A conjunction of Horn clauses is said to to be a *Horn formula*.

   As examples, $(\sim a \vee \sim b \vee c \vee d)$ is not a Horn clause since it has two unnegated literals. On the other hand, $(\sim a \vee \sim b \vee c)$ is a Horn clause, and $(\sim a \vee \sim b \vee c) \wedge (\sim c \vee a)$ is a Horn formula.

   It is known that a Horn formula can be checked for satisfiability in polynomial time (you are encouraged to find out how).

   Now consider the following variant of Horn clauses and formulae. A *relaxed Horn clause* is one that has two unnegated literals. The unnegated literals appearing in a relaxed Horn clause are said to be *paired*. A *relaxed Horn formula* φ is a CNF formula defined as follows:

   - Every clause is either a Horn clause or a relaxed Horn clause.
   - If u and v are paired literals in a relaxed Horn clause, then every clause C in φ satisfies the following properties.

     P1: C contains u iff it also contains v.

     P2: If $C \equiv (D \vee \sim u)$, then there exits a clause $C' \equiv (D \vee \sim v)$ in φ.

     P3: No clause contains both a literal and its negation.

   (a) *[10 marks]* Show that the satisfiability of a relaxed Horn formula can be in time polynomial in the size of the formula *[Hint: You need to first read up on how satisfiability of Horn formulas is checked in order to answer this question.]*

   (b) *[10 marks]* Suppose properties P1 and P2 above are not satisfied by clauses of φ. If there are k relaxed Horn clauses in φ, show (by providing an algorithm) that satisfiability of φ can be checked in time that is polynomial in n (size of the formula) and exponential in k.

2. In this question, we wish to state certain properties of natural numbers in first order logic. You may use the interpreted predicates $<$ and $=$, and interpreted functions $*$ and $+$ on natural numbers with the usual interpretations. You may also use $one()$ as an interpreted nullary function that returns the natural number 1.

   Give first order logic sentences expressing the following properties of natural numbers.

(a) *[10 marks]* There are natural numbers that cannot be expressed as one natural number raised to the power of another natural number different from 1.

*For this question, you are allowed to use additional **uninterpretted** function symbols (other than $*$ and $+$) in your answer φ. However, every structure that satisfies φ and interprets $+$, $*$, $<$ and $=$ as described above must necessarily assert the above property of natural numbers. In other words, it should not be possible to find a structure satisfying your answer φ unless the above property holds for natural numbers. Note the emphasis on additional functions being **uninterpretted** in the above requirement. Specifically, you are not allowed to use an interpretted exponentiation function directly.*

(b) *[5 marks]* For every natural number n, there are infinitely many other natural numbers such that the greatest common divisor of n and each of these other numbers is 1.

(c) *[5 marks]* There are infinitely many natural numbers that have only one way of factorizing them as the product of two natural numbers.