CS719 Graded Homework #2

Max Marks: 60 **Due date: Aug 30, 2010 (5 pm)**

- *Be brief, complete and stick to what has been asked.*
- *If needed, you may cite results/proofs covered in class without reproducing them.*
- ***Discussion among students is fine, but the solution you turn in must be your own solution in your own words. Cases of copying or indulgence in unfair means will be severely penalized, including award of FR grade.***

Problems on First Order Logic:

1. *[10 marks]* Consider the first order logic sentence:

$$\varphi \equiv (\forall x\,(P(x) \to \exists y\,Q(x,y))) \wedge (\forall x \forall y\,(Q(x,y) \to (P(x) \wedge P(y)))) \wedge (\exists x \exists y\,(Q(x,y) \wedge \forall z\,\neg Q(y,z)))$$

Show using resolution for first order logic that φ is unsatisfiable. Note that φ is not in Skolem Normal form. So you must first Skolemize it, find most general unifiers, and then use first order resolution to show that the formula is unsatisfiable.

2. *[15 marks]* In studying first order logic (FOL) and its applications, two students X and Y have entered into an argument over the expressive power of FOL sentences in describing properties of directed graphs. Formally, let $\Sigma = \{E, =\}$ be a vocabulary, where E is a binary predicate and $=$ is the usual equality relation.

   - X thinks that if a certain property of graphs cannot be expressed by any FOL sentence φ over Σ, then such a property cannot be expressed even by an infinite family Γ of FOL sentences over Σ. Although the student isn't sure of the exact argument, she thinks this should be provable using the Compactness Theorem of first order logic.
   - Y thinks that every property of graphs can be expressed by an infinite family Γ of FOL sentences over Σ. Although he doesn't have a proof yet, he intuitively feels that since there are no bounds on the cardinality of Γ, it should be possible to express every property of graphs using a sufficiently large number (possibly infinite) of suitably designed FOL sentences over Σ.

   Note that when we say that a FOL sentence φ (or a family of FOL sentences Γ) expresses a property $\mathcal{P}$ of directed graphs, we mean the following. For every directed graph G with property $\mathcal{P}$, the corresponding Σ-structure M_G is a model of φ (or a model of every FOL sentence in Γ). Similarly, every model M_G of φ (or of Γ) represents a graph G with property $\mathcal{P}$.

   You are required to show that both X and Y are wrong. In other words:

   (a) *[7.5 marks]* Show that there are properties of graphs that cannot be expressed by an individual FOL sentence on Σ, but can be expressed by an infinite family of FOL sentences on Σ.

   (b) *[7.5 marks]* Show that there are properties of graphs that cannot be expressed by any infinite family of FOL sentences on Σ.

3. *[25 marks]* Consider the first order logic sentence
$\varphi \equiv \exists x \forall y \exists z\ ((x = y) \vee (P(x,y,z) \wedge \sim P(y,x,z)))$, where P is a ternary predicate.

(a) *[5 marks]* Show that for every natural number $n \in \{1, 2, \ldots\}$, φ has a model of size n (the size of a model is the cardinality of its universe).

(b) *[5 marks]* Let $M = (U^M, P^M)$ and $N = (U^N, P^N)$ be arbitrary models of φ, where U^M and U^N are countably infinite. Show that there exists a model $M \times N = (U^M \times U^N, P^{M \times N})$ of φ.

(c) *[5 marks]* Can the result of part (ii) above be generalized to any first-order logic sentence ψ? Give brief justification for your answer.

(d) *[10 marks]* Now consider an arbitrary first order logic sentence η that ***does not use the equality predicate***. A logician claims that if η has a model of finite size $n > 0$, then η must also have a model of size $n + 1$. If you think the logician is correct, provide a proof of the claim. Otherwise, provide a counterexample, i.e. a sentence η and a natural number $n > 0$ such that η has a model of size n but no models of size $n + 1$.

4. *[5 + 5 marks]* Exercise 3.21 from "A First Course in Logic" by Shawn Hedman.