CS719 Graded Homework #3

Max Marks: 65 **Due date: Oct 19, 2010 (5 pm)**

- *Be brief, complete and stick to what has been asked.*
- *If needed, you may cite results/proofs covered in class without reproducing them.*
- ***Discussion among students is fine, but the solution you turn in must be your own solution in your own words. Cases of copying or indulgence in unfair means will be severely penalized, including award of FR grade.***

Problems on Partial Orders and Lattices:

1. *[5 + 5 marks]* Exercise 1.23 (i) and (ii) from Davey & Priestley's book.

2. *[6 + 3×3 marks]* Exercise 1.18(i) and Exercise 1.19 from Davey & Priestley's book. For Exercise 1.19, you need to give the cardinality of $\mathcal{O}(P)$ only for P_1, P_3 and P_6', where P_6' is obtained by extending the poset P_6 shown in the book, so that it has all chains containing upto n elements (without considering the top and bottom elements of P_6'). For example, P_6 contains chains of 1, 2, 3, 4 and 5 elements, without considering the top and bottom elements of P_6.

3. *[5 marks]* Let $(X, \leq)$ be a non-empty poset and let $P = \wp(X)$, i.e., the powerset of X. Define a relation $\ll$ on P as follows. For $A, B \in P$, $A \ll B$ iff for every $a \in A$, there exists $b \in B$ such that $a \leq b$. Prove that if $A \ll B$ and $B \ll A$, then A and B have the same set of maximal elements.

4. *[5+5+5 marks]* Exercise 2.17 from Davey & Priestley's book. Please answer all parts of this question.

5. *[5 marks]* An ideal I of a lattice $(L; \leq)$ is said to be *prime* if $I \subset L$ and $\forall a, b \in L,\ a \wedge b \in I \Rightarrow a \in I$ or $b \in I$. Show that every ideal of a lattice is prime if and only if L is a chain.

6. *[5 marks]* A convex subset S of a poset $(P; \leq)$ is one such that $\forall x, y \in S$, $\forall z \in P,\ x \leq z \leq y \Rightarrow x \in S$. Show that a nonempty subset S of lattice L is a convex sublattice of L if and only if $S = I \cap F$, where I is an ideal of L and F is a filter of L.

7. (a) *[5 marks]* Let L and M be lattices, where L has a bottom element 0, and let $f : M \to L$ be a lattice homomorphism. The *ideal kernel* of f, denoted $ker(f)$, is defined to be $f^{-1}(0) = \{x \mid x \in M, f(x) = 0\}$. Show that $ker(f)$ is an ideal of M (hence the name "ideal kernel").

   (b) *[5 marks]* Let $\mathbf{M_3} = (\{\bot, \top, a, b, c\}; \leq)$ be the lattice with covering relation: $\bot -\!\!< a, \bot -\!\!< b, \bot -\!\!< c, a -\!\!< \top, b -\!\!< \top, c -\!\!< \top$. Clearly, $\{\bot, a\}$ is an ideal of $\mathbf{M_3}$. Show that $\{\bot, a\}$ cannot be $ker(f)$ for any lattice homomorphism $f : \mathbf{M_3} \to L$, where L is a lattice with a bottom element.