CS719 Mid-Semester Examination (Autumn 2013)

Max marks: 80 **Time: 2 hours**

- *Be brief, complete and stick to what has been asked.*
- *Untidy presentation of answers, and random ramblings will be penalized by negative marks.*
- *Unless asked for explicitly, you may cite results/proofs covered in class without reproducing them.*
- ***If you need to make any assumptions, state them clearly.***
- ***Do not copy solutions from others. Penalty for offenders: FR grade.***

1. Consider the relational (i.e. no functions) vocabulary $\Sigma = \{<, =, C, D, P\}$, where $<, C$ and D are binary predicates, P is a unary predicate and $=$ is the usual identity (or equality) relation.

   We wish to restrict our discussion to Σ-structures in which the universe is the set $\mathbf{N}$ of natural numbers, and $<$ is interpreted as the usual less than relation over natural numbers. The predicates C, D and P can, however, be be assigned different interpretations in different Σ-structures of interest.

   Every Σ-structure M satisfying the above restrictions can be thought of as defining a directed (possibly multi-)coloured graph G as follows:

   - The set of vertices V^G of G is $\{i \mid i \in \mathbf{N},\ P^M(i) = True\}$.
   - The set of directed edges E^G is $\{(i, j) \mid i, j \in \mathbf{N},\ P^M(i) \wedge P^M(j) \wedge D^M(i, j)\}$
   - For every vertex i in the graph (i.e. for every $i \in \mathbf{N}$ such that $P^M(i)$ is true), we say that it is assigned colour j iff $C^M(i, j)$ is true. Note that a vertex may be assigned multiple colours or no colour at all in a Σ-structure.

   Similarly, every directed (possibly multi)-coloured graph G on a countable number of vertices can be thought of as encoded by a Σ-structure M as follows. In the following, we assume that every vertex in the graph G as well as every colour used to colour vertices is identified by a unique natural number.

   - For every $i \in \mathbf{N}$, $P^M(i) = True$ iff there is a vertex identified by i in G.
   - For every $i, j \in \mathbf{N}$, $C^M(i, j) = True$ iff $P^M(i) = True$ and vertex i is coloured by colour j.
   - For every $i, j \in \mathbf{N}$, $D^M(i, j) = True$ iff $P^M(i) = True$ and $P^M(j) = True$ and (i, j) is a directed edge in G.

   (a) *[15 marks]* Write a sentence φ over Σ such that the set of Σ-structures *satisfying the above restrictions and satisfying* φ are exactly the set of directed graphs over countable vertices in which (i) every vertex is coloured with exactly one colour, and (ii) no directed edge has the same colour on the vertices at its ends, and (iii) only finitely many colours are needed to colour all vertices in the graph.

   (b) *[15 marks]* We wish to find if it is possible to write a set (not necessarily a singleton set) of first order logic sentences over the vocabulary Σ such that every sentence in the set evaluates to $True$ over a Σ-structures iff the Σ-structure interpretted as a coloured directed graph (as described above) is one in which (i) every vertex is coloured with exactly one colour, and (ii) no directed edge has the same colour on the vertices at its ends, and (iii) only finitely many colours are needed to colour every vertex.

   Note that unlike in the previous sub-question, you are no longer required to choose a specific universe or a specific interpretation of the predicate $<$. Note also that you are also allowed to use an infinite set of first order logic sentences to define the desired class of graphs.

   If your answer is in the affirmative, describe the set of first order logic sentences. Otherwise, give a proof of impossibility.

2. Consider the vocabulary $\Sigma = \{=, D\}$, where D is a binary relation. As discussed in class (and also explained in the previous question), every Σ-structure represents a directed graph, and every directed graph can be represented as a Σ-structure. In each subquestion below, you are given two first order logic sentences φ_A and φ_B over the vocabulary Σ. You are required to either show (by any technique studied in class) that $\varphi_A \wedge \neg\varphi_B$ is unsatisfiable, or give a directed graph that serves as a model of φ_A but not as a model of φ_B.

   (a) *[10 marks]*
   $\varphi_A \equiv \forall x\, \exists y\, (((\neg(y = x) \wedge D(x, y)) \vee \neg D(y, x)) \wedge (D(x, x) \vee (D(x, y) \wedge \neg D(y, y))))$
   $\varphi_B \equiv \forall x\, \exists y\, \forall z\, (D(x, y) \vee (D(y, z) \wedge \neg D(x, z)))$

   (b) *[10 marks] (Turned out to be trivial due to a typo in φ_B)*
   $\varphi_A \equiv \forall x\, \exists y\, \exists z((D(x, y) \vee D(x, z)) \wedge (\neg D(x, y) \vee D(y, z)) \wedge (\neg D(x, z) \vee D(z, y)))$
   $\varphi_B \equiv \forall x\, \forall y\, (D(x, y) \vee \neg D(x, y))$

3. We have seen in class that we can reduce satisfiability checking of an arbitrary first-order logic sentence φ to satisfiability checking of a FOL sentence $\varphi^\star$ (Skolem Normal Form of φ) with only universal quantifiers in PCNF, such that there is a model of φ with universe U iff there is a model of $\varphi^\star$ with universe U. In this question, we wish to investigate the possibility of a few other such simplifications.

   (a) *[15 marks]* Is it possible to reduce satisfiability checking of an arbitrary FOL sentence φ to satisfiability checking of an FOL sentence $\widehat{\varphi}$ in PCNF with no function symbols (but with possibly existential quantifiers), such that that there is a model of φ with universe U iff there is a model of $\widehat{\varphi}$ with universe U?

   If so, describe clearly the steps how you would transform an arbitrary FOL sentence φ with function symbols to the corresponding equisatisfiable sentence $\widehat{\varphi}$ in PCNF without function symbols. Otherwise, give justification for the impossibility of such an equisatisfiability reduction by a deterministic algorithm. Answers without justification will fetch 0 marks.

   (b) *[15 marks]* Is it possible to reduce satisfiability checking of an arbitrary FOL sentence φ to satisfiability checking of an FOL sentence $\widetilde{\varphi}$ in PCNF with no function symbols *and* also no existential quantifiers, such that that there is a model of φ with universe U iff there is a model of $\widetilde{\varphi}$ with universe U?

   If so, indicate clearly the steps how you would transform an arbitrary FOL sentence φ with function symbols to the corresponding equisatisfiable sentence $\widetilde{\varphi}$ in PCNF without function symbols and existential quantifiers. Otherwise, give justification for the impossibility of such an equisatisfiability reduction by a deterministic algorithm. Answers without justification will fetch 0 marks.