CS719 Practice Problems (Autumn 2013)

- ***These problems will not be graded.***
- ***Mutual discussion and discussion with the instructor/TA is strongly encouraged.***

1. *[From HW1, Autumn 2011]* Use the proof system of first order logic studied in class to prove each of the following sequents. You must indicate which proof rule you are applying at each step.

   (a) $\forall x \forall y. ((P(x) \vee Q(y) \wedge (Q(x) \vee \neg P(y)) \ \vdash \ \forall x. Q(x)$

   (b) $\vdash \ \exists y. ((\forall x. P(x)) \ \rightarrow \ P(y))$

2. *[From HW1, Autumn 2011]* We have seen in class that first-order sentences on a vocabulary consisting of a single binary predicate E (and possibly also using equality) can be used to describe several interesting properties of graphs. Show that there exists a first-order sentence ϕ on the vocabulary $\{E\}$ (and possibly also using equality) that characterizes each of the following classes of graphs. Thus, ϕ should be such that every graph belonging to the class under consideration is a model of ϕ, and every model of ϕ can be viewed as a graph belonging to the class under consideration.

   (a) Rooted trees with depth (i.e. no. of edges along any path from the root to a leaf) ≤ 3. Assume that each edge in a tree points from a node to its children.

   (b) Directed acyclic graphs with diameter (i.e. length of the longest directed path between any pair of vertices) ≤ 3.

   (c) Graphs with no vertex covers of size ≤ 3.

   Taking cue from the formulas you have written as part of your solution for the previous sub-problems, either show that the following sets of graphs are inexpressible in first-order logic over the vocabulary consisting of a binary predicate and possibly equality, or give a first-order logic formula characterizing the corresponding set of graphs.

   (a) The set of all finite rooted trees.

   (b) The set of all infinite rooted trees.

   (c) The set of all finite or infinite rooted trees.

3. *[From Quiz1, Autumn 2011]* Consider the following predicate logic sentences (formulae with no free variables):

   $\phi \ = \ \forall x(\exists y(E(x,y) \wedge \neg E(y,x)))$
   $\psi \ = \ \forall x(\forall y(\forall z((E(x,y) \wedge E(y,z)) \rightarrow \ E(x,z))))$.

   We wish to evaluate these sentences (i.e., determine their truth value) over models obtained from directed graphs. Given a directed graph, $G = (V_G, E_G)$, a model $\mathcal{M}_G$ is obtained by letting the domain of variables be V_G, and by letting predicate $E(x, y)$ evaluate to true if and only if there is a directed edge from x to y in E_G. Since there are no function symbols and only one binary predicate symbol E in ϕ and ψ, every directed graph uniquely defines a model over which ϕ and ψ can be evaluated.

(a) Give two directed graphs G_1 and G_2 with no more than 5 vertices in each such that ϕ evaluates to True over $\mathcal{M}_{G_1}$, and $\phi \rightarrow \psi$ evaluates to False over $\mathcal{M}_{G_2}$.

(b) We now wish to find a directed graph G such that $\phi \wedge \psi$ evaluates to True over $\mathcal{M}_G$. Indicate with justification whether G can be a finite graph (i.e., graph with finite number of vertices) and yet cause $\phi \wedge \psi$ to evaluate to True over $\mathcal{M}_G$.

4. *[From Quiz1, Autumn 2011]* Let Σ be a relational signature (i.e., no function symbols). Two Σ-structures M_1 and M_2 are said to be isomorphic iff there exists a bijection $f : U^{M_1} \rightarrow U^{M_2}$ such that for every k-ary predicate P in Σ and every k-tuple $(a_1, a_2, \ldots a_k) \in (U^{M_1})^k$ and $(b_1, b_2, \ldots b_k) \in (U^{M_2})^k$, we have the following: $P^{M_1}(a_1, a_2, \ldots a_k) = P^{M_2}(f(a_1), f(a_2), \ldots f(a_k))$ and $P^{M_2}(b_1, b_2, \ldots b_k) = P^{M_1}(f^{-1}(b_1), f^{-1}(b_2), \ldots f^{-1}(b_k))$. It can be shown that if M_1 and M_2 are isomorphic Σ-structures, then for every first-order logic sentence ϕ on the signature Σ, $M_1 \models \phi$ iff $M_2 \models \phi$.

   Now consider $\Sigma = \{=\}$, i.e., the signature containing only the equality predicate. We wish to show that there does not exist any first-order logic sentence ϕ over Σ such that $M \models \phi$ iff M has an even number of elements in its universe.

   Suppose, by way of contradiction, that there indeed existed such a formula ϕ.

   (a) Show using the Compactness Theorem and other arguments as appropriate that we must then have countably infinite Σ-structures M_1 and M_2 such that $M_1 \models \phi$ and $M_2 \models \neg\phi$.

   (b) Argue that both M_1 and M_2 must satisfy ϕ as well as $\neg\phi$. This is of course a contradiction; hence such a formula ϕ on the signature Σ cannot exist.

5. *[From Endsem, Autumn 2010]*

   Let $\psi = \forall x_0 (\forall x_1 \exists x_2 \forall x_3 \exists x_4 \, (\phi_1(x_1, x_3, x_4) \vee \phi_2(x_0, x_2, x_4)) \wedge \exists x_1 \forall x_2 \forall x_3 \exists x_4 \, (\phi_3(x_2, x_3, x_4) \vee \phi_4(x_0, x_1)))$, where ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are *quantifier-free* predicate logic subformulae. You are also told that the ϕ_i's are syntactically constructed by applying propositional connectives on k ternary predicates $P_1, P_2, \ldots P_k$ and m binary predicates $Q_1, Q_2, \ldots Q_m$. Moreover, none of the ϕ_i's have any function symbols.

   A smart logician now claims that she has an algorithm that can take arbitrary three-argument formulae ϕ_1, ϕ_2, ϕ_3 and an arbitrary two-argument formula ϕ_4 subject to the restrictions in the previous paragraph, and can determine the validity of ψ given above. If you think the logician is right, briefly describe how you would go about algorithmically checking the validity of ψ (you could give a pseudo-code, for example). If you think the logician is incorrect, describe your reasons for the same.

6. *[From Endsem, Autumn 2010]* Let f be a unary function and $\mathcal{M}$ be a model containing an interpretation of f. In general, some elements x in the universe (or domain) of $\mathcal{M}$ satisfy $f(x) = x$, while other elements satisfy $\neg(f(x) = x)$.

(a) We wish to write a predicate logic sentence ϕ that evaluates to True in $\mathcal{M}$ if and only if the universe of $\mathcal{M}$ has more elements x satisfying $\neg(f(x) = x)$ than elements y satisfying $f(y) = y$. Is it possible to have such a predicate logic sentence ϕ? If your answer is in the affirmative, you must provide ϕ. If your answer is in the negative, you must prove that such a sentence ϕ cannot exist.

(b) Suppose we now wish to obtain a predicate logic sentence ψ that evaluates to True in $\mathcal{M}$ if and only if there are infinitely many elements x in the universe of $\mathcal{M}$ that satisfy $f(x) = x$. Is it possible to have such a predicate logic sentence ψ? If your answer is in the affirmative, you must provide ψ. If your answer is in the negative, you must prove that such a sentence ψ cannot exist.

7. *[From Endsem, Autumn 2010]* Consider the predicate logic sentence $\phi = \phi_1 \wedge \phi_2 \wedge \phi_3$, where

$$\begin{aligned}
\phi_1 &= \forall x\,(f(g(x)) = g(x)),\\
\phi_2 &= \forall x\,(g(f(x)) = f(x)),\\
\phi_3 &= \forall x \exists y\,((x = g(y)) \vee (x = f(y)))
\end{aligned}$$

(a) Let $\mathcal{H}$ be the Herbrand Universe for the sentence ϕ. Let $M_{\mathcal{H}}$ be a model with the universe (or domain) being $\mathcal{H}$, and with the obvious interpretations of functions f and g, i.e., if t is a term in $\mathcal{H}$, the terms $f(t)$ and $g(t)$ in $\mathcal{H}$, are defined to be the result of applying f and g respectively to t. If $M_{\mathcal{H}} \models \phi$, how many distinct elements can be present in the universe of $M_{\mathcal{H}}$.

(b) Let $\phi_4 = \exists x\,\neg(f(x) = g(x))$. Show using Herbrand's Theorem that $\phi_1 \wedge \phi_2 \wedge \phi_3 \wedge \phi_4$ is unsatisfiable.

8. *[From Endsem, Autumn 2011]* Let ϕ be a formula in first order (FO) logic, such that ϕ has at least one existential quantifier. Let $\phi^\star$ be the formula obtained by transforming ϕ to Skolem Normal Form (SNF).

(a) Give an example of ϕ over the vocabulary $\{=\}$ such that both ϕ and $\neg\phi^\star$ are satisfiable but $\phi \wedge \neg\phi^\star$ is unsatisfiable.

(b) Give an example of ϕ over the vocabulary $\{=\}$ such that $\phi \wedge \neg\phi^\star$ is satisfiable. Give also a model of $\phi \wedge \neg\phi^\star$.

(c) Suppose ϕ_1 and ϕ_2 are FO formulae over the vocabulary $\{=\}$, each containing existential quantifier(s), such that:

i. Each of ϕ_1, ϕ_2, $\neg\phi_1^\star$ and $\neg\phi_2^\star$ are satisfiable.

ii. Each of $\phi_1 \wedge \neg\phi_1^\star$ and $\phi_2 \wedge \neg\phi_2^\star$ are unsatisfiable.

Show that $\phi_1 \wedge \phi_2$ must be satisfiable.

9. *[From Endsem, Autumn 2011]* Given below is a collection of first order sentences over the relational vocabulary $\{P_1, P_2, G\}$.

ϕ_1	$\forall x\, \forall y.\, (P_1(x) \to (G(x,x) \vee \neg G(x,y)))$
ψ_1	$\forall x\, \forall y.\, (P_2(x) \to (G(x,x) \vee \neg G(x,y)))$
ϕ_2	$\forall x.\, (G(x,x) \to P_1(x))$
ψ_2	$\forall x.\, (G(x,x) \to P_2(x))$
ϕ_3	$\forall x.\, (P_1(x) \vee \exists z\, .G(x,z))$
ψ_3	$\forall x.\, (P_2(x) \vee \exists z\, .G(x,z))$
α	$\exists x.\, \neg(P_1(x) \leftrightarrow P_2(x))$

Show using first-order resolution that the conjunction of these sentences is unsatisfiable.

You must use the following table to present your resolution proof. Each row in this table has four columns: a numerical clause id, the clause from an SNF representation, how this clause was obtained, and substitutions used (if any) to obtain the clause. A couple of rows of the table have been filled to illustrate its usage. The third column should contain one of the following comments: "Given" (if the clause is given) or "Resolvent of i, j" (if the clause is obtained by resolving clauses with ids i and j). In the latter case, you must also indicate the substitution used for resolving in the final column.

ID	Clause	Genesis	Substitution (if any)
1	$\neg P_1(x) \vee G(x,x) \vee \neg G(x,y)$	Given (from ϕ_1)	–
2	$\neg G(a,a) \vee P_1(a)$	Given (from ϕ_2)	–
3	$\neg P_1(x) \vee \neg G(x,y) \vee P_1(x) \equiv \mathsf{True}$	Resolvent of 1, 2	$a \mapsto x$ in 2

10. *[From Endsem, Autumn 2011]* Consider an employee relational database containing three tables R_1, R_2 and R_3, each with two columns. The columns of R_1 are labeled "Name" and "ID", those of R_2 are labeled "ID" and "Dept", while those of R_3 are labeled "Name" and "Name". Table R_1 gives the association between names and IDs of employees, table R_2 gives the association between IDs and departments, while table R_3 gives the association between employees and their supervisors ($(x,y) \in R_3$ means x is a supervisor of y). Note that R_3 **may not** be a transitive relation. You may however assume that each name is associated with a unique ID, and each ID is associated with at least one department.

    Every instance D of such a database can be viewed as a first-order structure M_D over the relational vocabulary $\Sigma : \{=, R_1, R_2, R_3, \mathsf{Name}, \mathsf{ID}, \mathsf{Dept}\}$, where R_1, R_2 and R_3 are binary predicates and Name, ID and Dept are unary predicates. The universe of M_D is the set of all names, IDs and departments in D. The interpretation of the predicate Name in M_D is such that $\mathsf{Name}(x)$ evaluates to True iff x is a name in D. The interpretations of predicates ID and Dept in M_D are similar. The interpretation of R_1 in M_D is such that $R_1(x,y)$ evaluates to True iff x is a name, y is an ID, and there is a row in table R_1 of D that contains the tuple (x,y). The interpretations of R_2 and R_3 in M_D are similar.

    A first-order formula over Σ can now be viewed as a query over the corresponding relational database. For example, given an instance D of the database, the formula $\varphi(x) \equiv (\mathsf{Dept}(x) \wedge \exists y.\, (\mathsf{ID}(y) \wedge R_2(y,x)))$ corresponds to a query that returns all departments in D with at least one ID associated with them.

For each of the following queries described in English, either give a first-order formula over the vocabulary Σ or prove that such a formula cannot exist.

(a) Find all departments in which every employee is her own supervisor.

(b) An employee x is said to be *isolated* from a department y iff x doesn't belong to y, and x is not related to anybody belonging to y in the transitive closure of R_3. Find all employees who are isolated from some department.

11. Consider the vocabulary $\Sigma = \{=, f, c\}$, where $=$ is the usual equality predicate, f is a unary function and c is a constant (or null-ary function). Clearly, there are infinitely many Σ-structures. A subset S of these structures is said to be first-order definable iff there is a first-order logic sentence φ over Σ such that the set of models of φ is exactly S.

    (a) Describe a subset S of Σ-structures such that S contains at least one Σ-structure of every (finite and infinite) non-zero cardinality, but S is not first-order definable. You must prove why the subset S you have chosen is not first-order definable using Σ.

    (b) Give a first-order logic sentence φ over Σ such that φ has models of every (finite and infinite) non-zero cardinality, and for every cardinality greater than 1, there are exactly two non-isomorphic models of φ.

12. (a) Consider the following procedure that attempts to determine the satisfiability of a first-order logic sentence in SNF.

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 i = 1;

 while (true)
   Check unsatisfiability of the set of all ground clauses
     using terms of length at most i;
   If unsatisfiability detected, return UNSAT;

   Check satisfiability of the formula using all structures
     on the vocabulary with at most i elements in the universe;
   If satisfiablity detect, return SAT;

   i++;
// end while
```

    Give an example of a first-order logic sentence for which the above procedure fails to terminate.

    (b) Is it possible to have *a set of first-order sentences* such that the only models of the entire set of sentences have uncountable universes? If so, give an example. Otherwise, prove why this is not possible.