CS719 End-Semester Examination

Max Marks: 75 **Time: 3 hours**

- ***Please be brief in your answers. None of the questions requires lengthy solutions. Unnecessarily lengthy and rambling solutions will incur negative marks.***
- ***Do not reproduce proofs covered in class – if needed you can simply cite them.***
- ***If you need to make any assumptions, state them very clearly.***
- ***Do not indulge in copying or unfair means during the examination. Offenders will be awarded an FR grade.***

1. Let $(L; \vee_L, \wedge_L)$ be a lattice with DCC. Let $p : L \rightarrow \{\mathsf{True}, \mathsf{False}\}$ be a predicate such that $\forall x, y \in L,\ p(x \vee_L y) \Rightarrow p(x) \vee p(y)$ (note that $\vee$ represents Boolean disjunction, and is distinct from $\vee_L$). Define $S = \{x \mid x \in L,\, p(x) = \mathsf{True}\}$, and $S' = \{x \mid x \in \mathcal{J}(L),\, p(x) = \mathsf{True}\}$.

   (a) Prove the following:

      i. *[5 marks]* $S = \emptyset$ if $S' = \emptyset$.

      ii. *[5 marks]* $S = L$ if $S' = \mathcal{J}(L)$.

      iii. *[5 marks]* $|L \setminus S|$ is finite if $|\mathcal{J}(L) \setminus S'|$ is finite.

   (b) *[3+3+4 marks]* Do the above results hold if L has ACC instead of DCC? In each case, you must either give a proof or provide a counterexample.

2. A lattice L with bottom element $\perp$ is said to be *pseudocomplemented* if for each $a \in L$, there exists a unique $a^* \in L$, such that (i) $a^* \wedge a = \perp$, and (ii) $b \leq a^*$ for every $b \in L$ that satisfies $b \wedge a = \perp$. In other words, the set $\{b \mid b \in L,\, b \wedge a = \perp\}$ has a unique maximum element, for every $a \in L$. .

   (a) *[5 marks]* Show that every finite distributive lattice is pseudocomplemented.

   (b) *[3 marks]* Give an example of a distributive lattice that is pseudocomplemented but is not complemented.

   (c) *[7 marks]* Give two non-isomorphic distributive lattices that are not pseudocomplemented. Your answer to this subquestion must clearly identify the two lattices, and argue that they are non-isomorphic but distributive. In addition, you must show why your lattices are not pseudocomplemented.

3. *[10 marks]* Given a first-order vocabulary $\mathcal{V}$, let $\mathcal{M}_{\mathcal{V}}$ denote the set of all $\mathcal{V}$-structures, i.e. universe of elements and interpretation of each predicate and function symbol in $\mathcal{V}$. Let $\Phi : \mathcal{M}_{\mathcal{V}} \rightarrow \{\mathsf{True}, \mathsf{False}\}$ and $\Psi : \mathcal{M}_{\mathcal{V}} \rightarrow \{\mathsf{True}, \mathsf{False}\}$ be predicates on $\mathcal{V}$-structures such that

   - Φ is closed under sub-structures. Thus, for all $M_1, M_2 \in \mathcal{M}_{\mathcal{V}}$, if $\Phi(M_1) = \mathsf{True}$ and M_2 is a sub-structure of M_1, then $\Phi(M_2) = \mathsf{True}$.
   - Ψ is expressible in first-order logic. Thus, there exists a (possibly infinite) set Γ of first-order sentences on the vocabulary $\mathcal{V}$, such that $M \models \Gamma$ iff $\Psi(M) = \mathsf{True}$.

- For all finite $M \in \mathcal{M}_{\mathcal{V}}$, $\Phi(M) \Rightarrow \Psi(M)$. Thus, every $\mathcal{V}$-structure with finite universe that satisfies Φ also satisfis Ψ.

Show using the Compactness Theorem for first-order logic that $\Phi(M) \Rightarrow \Psi(M)$ for all $M \in \mathcal{M}_{\mathcal{V}}$, i.e. the implication holds even when M's universe is infinite.

4. (a) *[5 marks]* Use the Nelson-Oppen algorithm to decide the satisfiability of the quantifier-free formula obtained by conjoining the following clauses. Note that the resulting formula is in the mixed theories of equality with uninterpretted functions (EUF) and linear arithmetic over reals. You must indicate each step of application of the Nelson-Oppen algorithm, show all equalities that are inferred and propagated between the two theory solvers, and indicate the final result (satisfiable/not satisfiable).

No.	Clause	No.	Clause
C_1	$g(f(g(x))) \neq g(f(x))$	C_2	$g(x) = x + y$
C_3	$f(x) = x - z$	C_4	$z \geq 0$
C_5	$y \leq 0$	C_6	$z - y \leq 0$

(b) *[5 marks]* Use Herbrand's Theorem to show that the following first-order sentence is unsatisfiable. You must clearly indicate all your steps. Merely arriving at the right answer by the wrong means will fetch no marks.

$$(\forall x.\, g(x) = g(g(x))) \wedge (\forall x \exists y.\, x = g(y)) \wedge (\exists x.\, g(x) \neq x)$$

5. *[7.5 marks]* Let $L = (A; \leq)$ be a finite lattice and $L^\partial = (A; \geq)$ be its order-dual, i.e. the lattice obtained by considering the dual of the partial order $\leq$ on the set A. Show that if $L \cong L^\partial$ but $\mathcal{J}(L) \not\cong (\mathcal{M}(L))^\partial$, then L cannot be a distributive lattice.

6. (a) Let E be a binary predicate. We have seen in class that an $\{E\}$-structure M can be interpretted as a graph G_M, where the universe of M corresponds to the vertices in G_M, and the interpretation of E in M gives the edge relation in G_M.

   i. *[3 marks]* We wish to find a first order logic (with equality) sentence φ on the vocabulary $\{E\}$ such that $M \models \varphi$ iff G_M is a collection of (not necessarily disjoint) triangles. If you think this is possible, give φ and briefly explain why models of φ represent a collection of triangles. Otherwise, prove that it is not possible to find such a φ.

   ii. *[4.5 marks]* Show that it is not possible to obtain a (possibly infinite) set Γ of first-order sentences (with equality) on the vocabulary $\{E\}$ such that $M \models \Gamma$ iff G_M contains at least one cycle.