CS719 Mid-Semster Examination

Max marks: 50 **Time: 2 hours**

- *Be brief, complete and stick to what has been asked.*
- *If needed, you may cite results/proofs covered in class without reproducing them.*
- ***If you need to make any assumptions, state them clearly.***
- ***Do not copy solutions from others or indulge in unfair means.***

1. *[5 + 5 + 5 marks]*

   Let $(L; \wedge, \vee)$ be a lattice. An ideal I of L is called *proper* if $I \neq L$. Furthermore, ideal I is called *prime* if I is proper, and for every $a, b \in L$, $a \wedge b \in I$ implies $a \in I$ or $b \in I$. Let $Spec(L)$ (also called the spectrum of L) denote the set of prime ideals of L.

   (a) Let $f : L \to M$ be a lattice homomorphism.

      i. Show that if P is an ideal of M, then $f^{-1}(P)$ is an ideal of L.

      ii. Show that if $P \in Spec(M)$, then $f^{-1}(P) \in Spec(L)$.

   (b) For this subquestion, assume that L is a bounded lattice (i.e. has $\top$ and $\bot$), and I is a non-empty prime ideal of L. Show that for every $a \in L$, if a has a complement $a' \in L$, then exactly one of a or a' is in I.

2. *[5 + 5 marks]* Let P and Q be partially ordered sets, and $f : P \to Q$ be a map.

   (a) Prove that f is monotone if and only if the induced inverse map satisfies $f^{-1}(D) \in \mathcal{O}(P)$ for all $D \in \mathcal{O}(Q)$.

   (b) Assuming f to be monotone, and $f^{-1} : \mathcal{O}(Q) \to \mathcal{O}(P)$ to be the induced inverse map (as in the previous subquestion), show that f is surjective if and only f^{-1} is injective.

3. *[5 + 5 + 5 marks]* Let $A = \{a_1, a_2, \ldots\}$ and $B = \{b_1, b_2, \ldots\}$ be disjoint countably infinite sets. Define a relation R on $A \cup B$ as follows:
   $a_{i+1} \ R \ a_i$, $\quad b_{i+1} \ R \ b_i$, $\quad$ and $b_i \ R \ a_i$ for all $i \in \{1, 2, \ldots\}$.

   (a) Show that R is a covering relation.

   (b) Show that the corresponding partially ordered set $L = (A \cup B; R)$ is a lattice.

   (c) What are the join-irreducible elements of L? Does the set of join-irreducible elements form a join-dense subset of L? Give justification for your answer.

4. *[5 + 5 marks]*

   (a) Let L be a lattice. Prove that $I \subseteq L$ is an ideal of L if and only if I is a sublattice and that $i \in I, a \in L$ implies $a \wedge i \in I$.

   (b) Let L be a lattice. Show that if $a \in L$, then the *join map* $j_a : L \to L$ defined by $j_a x = x \vee a$ is monotone. Indicate with justification if it is also an order embedding.