CS781:
A Quick Primer on Abstract Interpretation for Neural Networks

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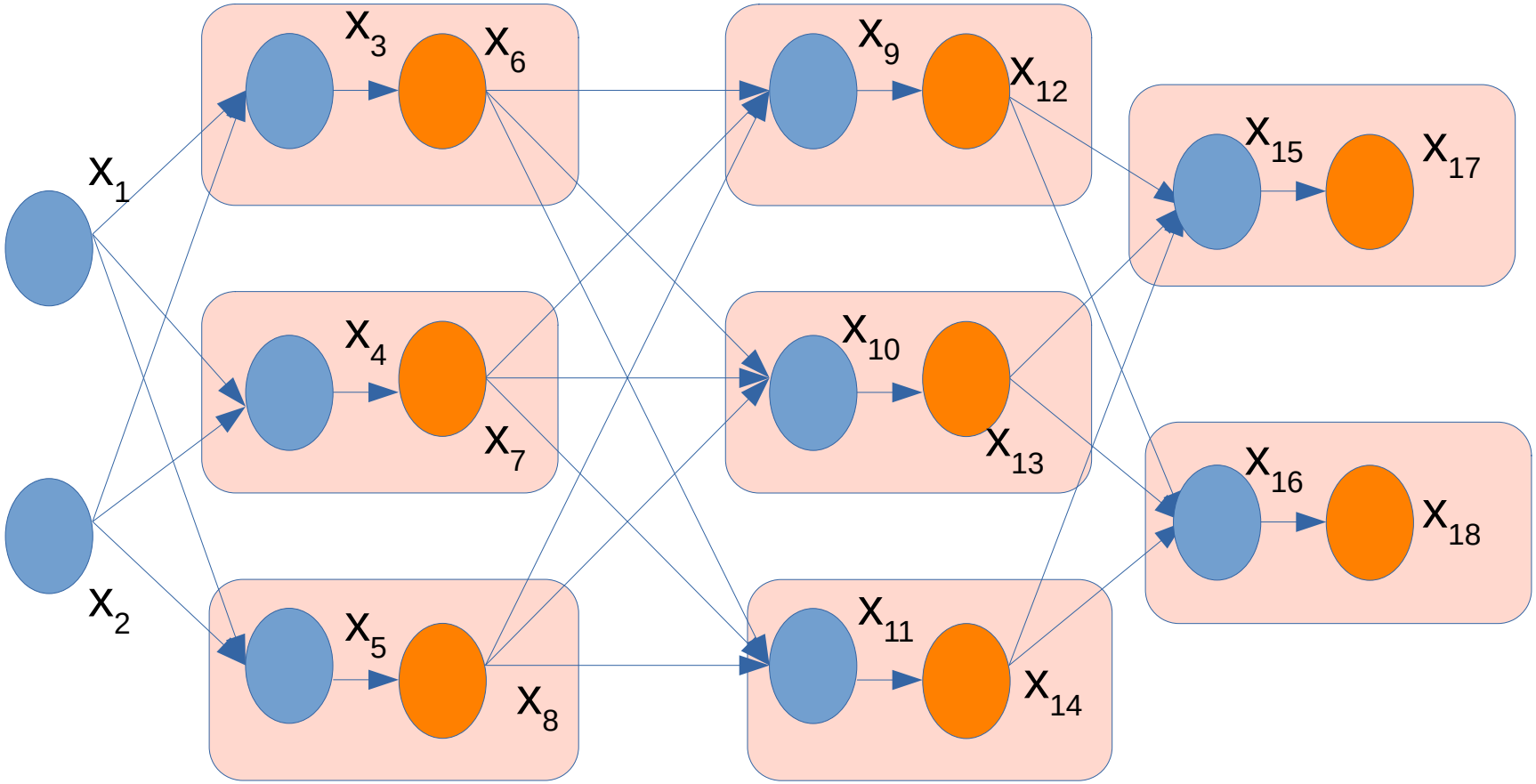
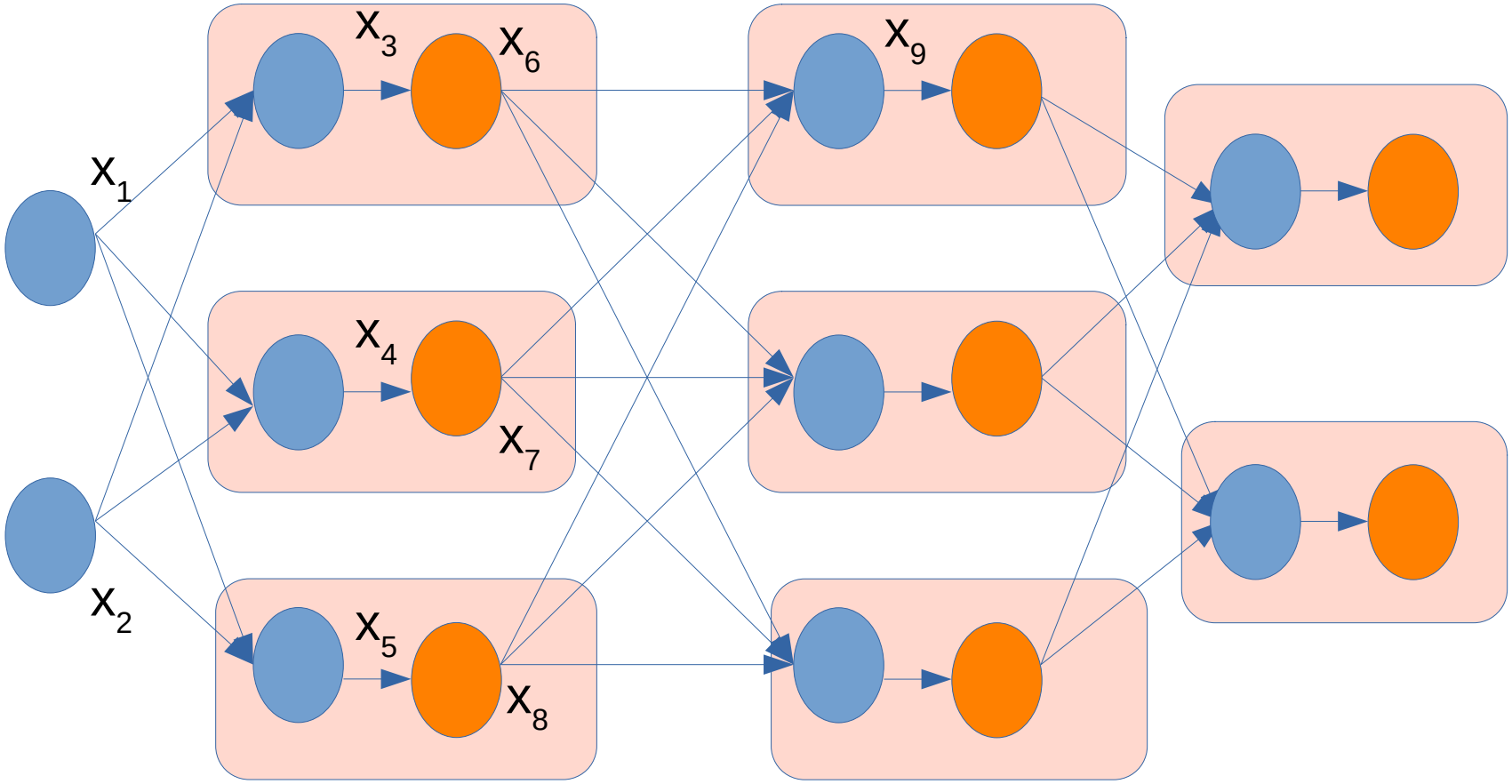
Notion of State in Neural Network



State: $(x_1, x_2, \ldots x_{18})$ in R^{18}

State Change in Feed-Forward Neural Network

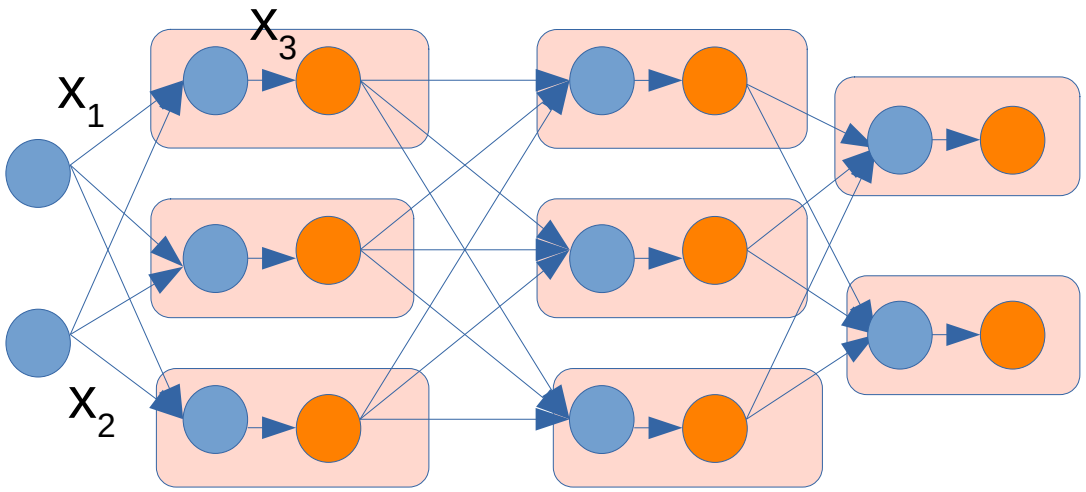


$(x'_1, x'_2, \ldots x'_{17}, x'_{18}) = f(x_1, x_2, \ldots x_{18})$, where

x'_i = new value of x_i obtained as per neuron i's functionality;

$x'_1 = x_1$, and $x'_2 = x_2$ (inputs assumed to be stable)

State Change in Feed-Forward NN as a sequence of instrns



$(x'_1, x'_2, x'_3, \ldots x'_{18}) = f(x_1, x_2, x_3, \ldots x_{18});$

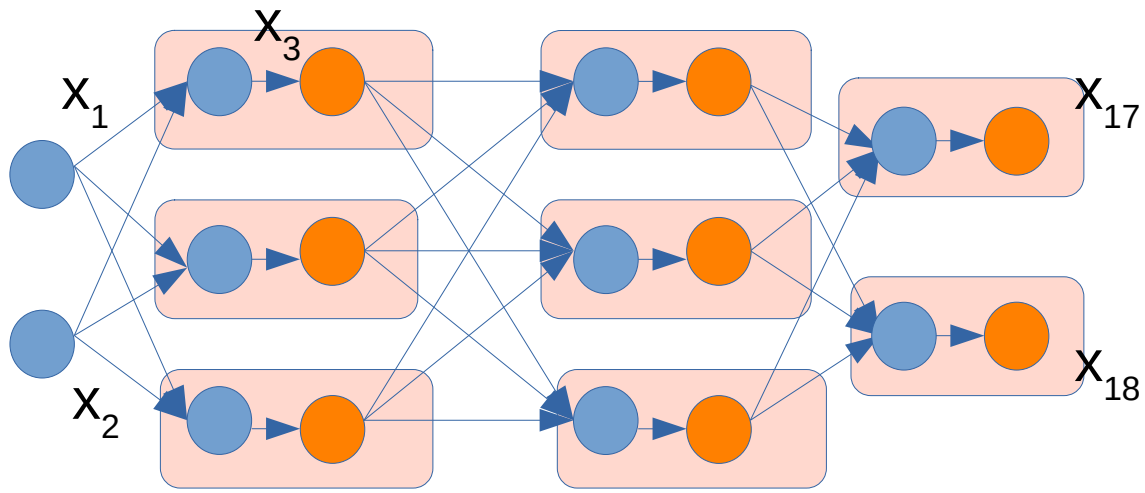
$(x''_1, x''_2, x''_3, \ldots x''_{18}) = f(x'_1, x'_2, x'_3, \ldots x'_{18});$

… repeat 18 times (at most)

NN computation: a sequence of state transitions caused by seq of instructions

Proving Property of a FF NN

Pre-condition on (x1, x2)



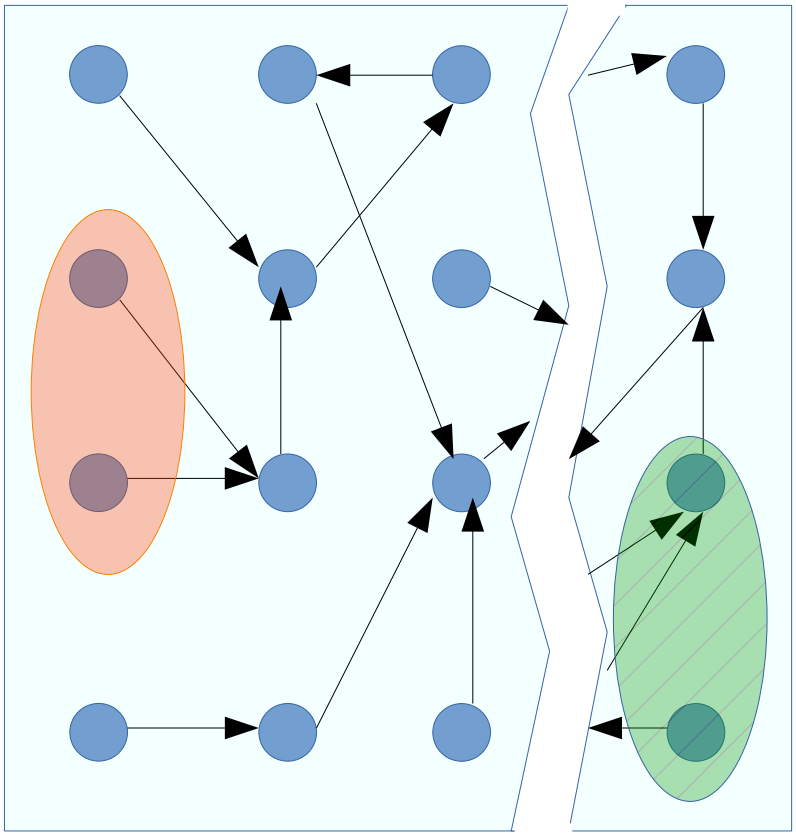
Post-condition on (x17, x18)

{Pre-condition on (x1, x2)}

$$(x'_1, x'_2, \ldots x'_{18}) = f(x_1, x_2, \ldots x_{18});$$
$$(x''_1, x''_2, \ldots x''_{18}) = f(x'_1, x'_2, \ldots x'_{18});$$
$$\ldots$$

{Post-condition on (x17, x18) }

NN Computation as a State Transition System



{Pre-condition on (x1, x2)}

$(x'_1, x'_2, \ldots x'_{18}) = f(x_1, x_2, \ldots x_{18});$

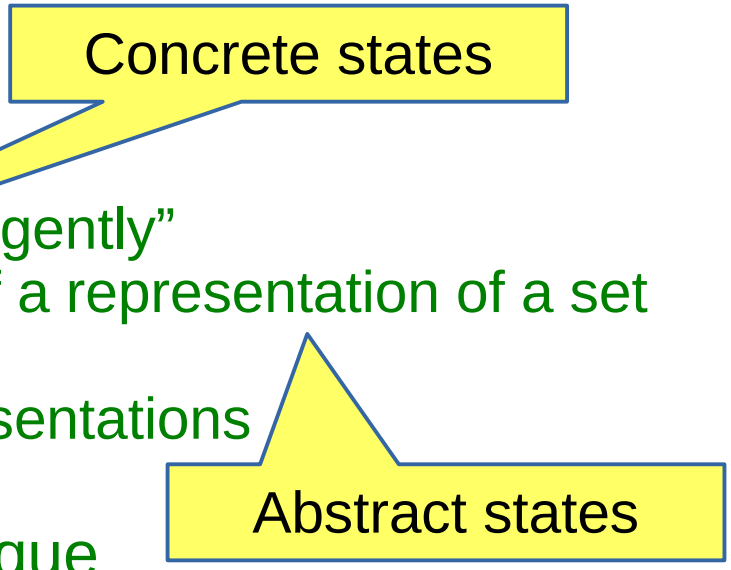
$(x''_1, x''_2, \ldots x''_{18}) = f(x'_1, x'_2, \ldots x'_{18});$

...

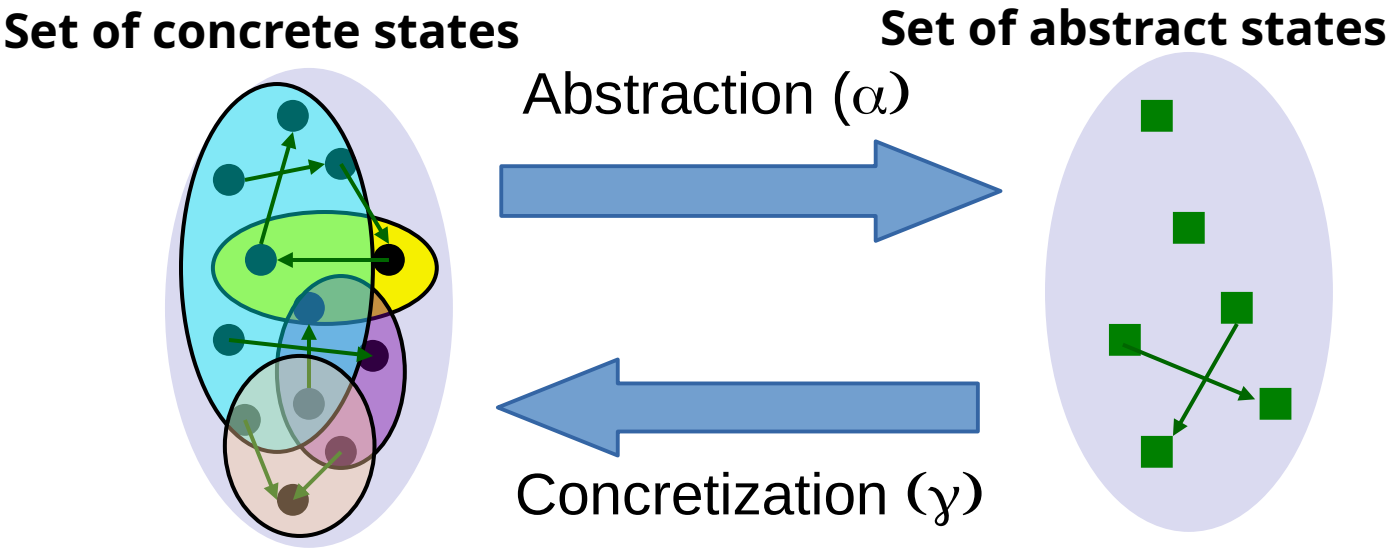
{Post-condition on (x17, x18) }

Dealing with State Space Size

- Infinite state space
  - Difficult to represent using state transition diagram
  - Can we still do some reasoning?
- Solution: Use of abstraction
  - Naive view
    - Bunch sets of states together "intelligently"
    - Don't talk of individual states, talk of a representation of a set of states
    - Transitions between state set representations
  - Granularity of reasoning shifted
  - Extremely powerful general technique
    - Allows reasoning about large/infinite state spaces

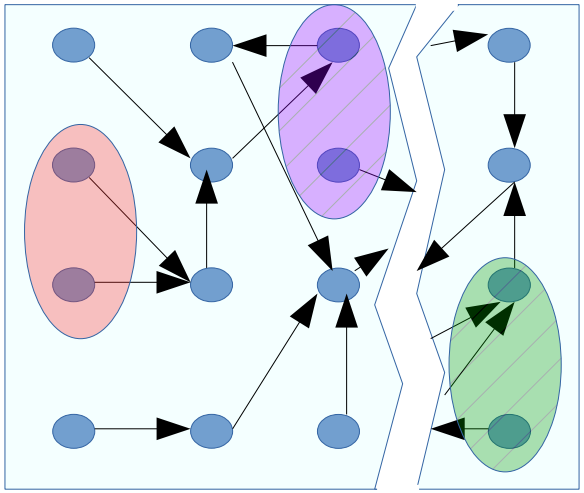


A Generic View of Abstraction

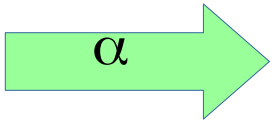


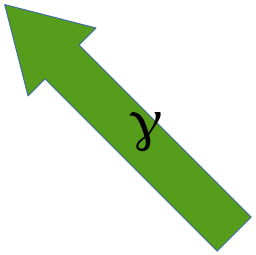
- Every subset of concrete states mapped to unique abstract state
- Desirable to capture containment relations
- Transitions between state sets (abstract states)

The Game Plan

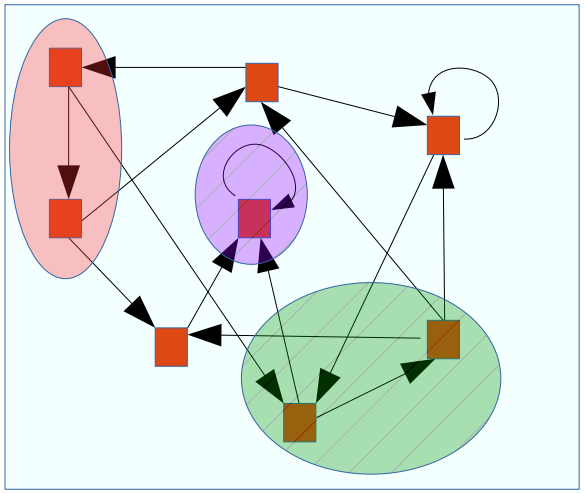


CONCRETE STATES

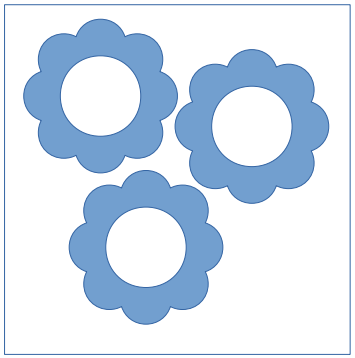




ABSTRACT STATES



Yes,
Proof



No,
Counterexample

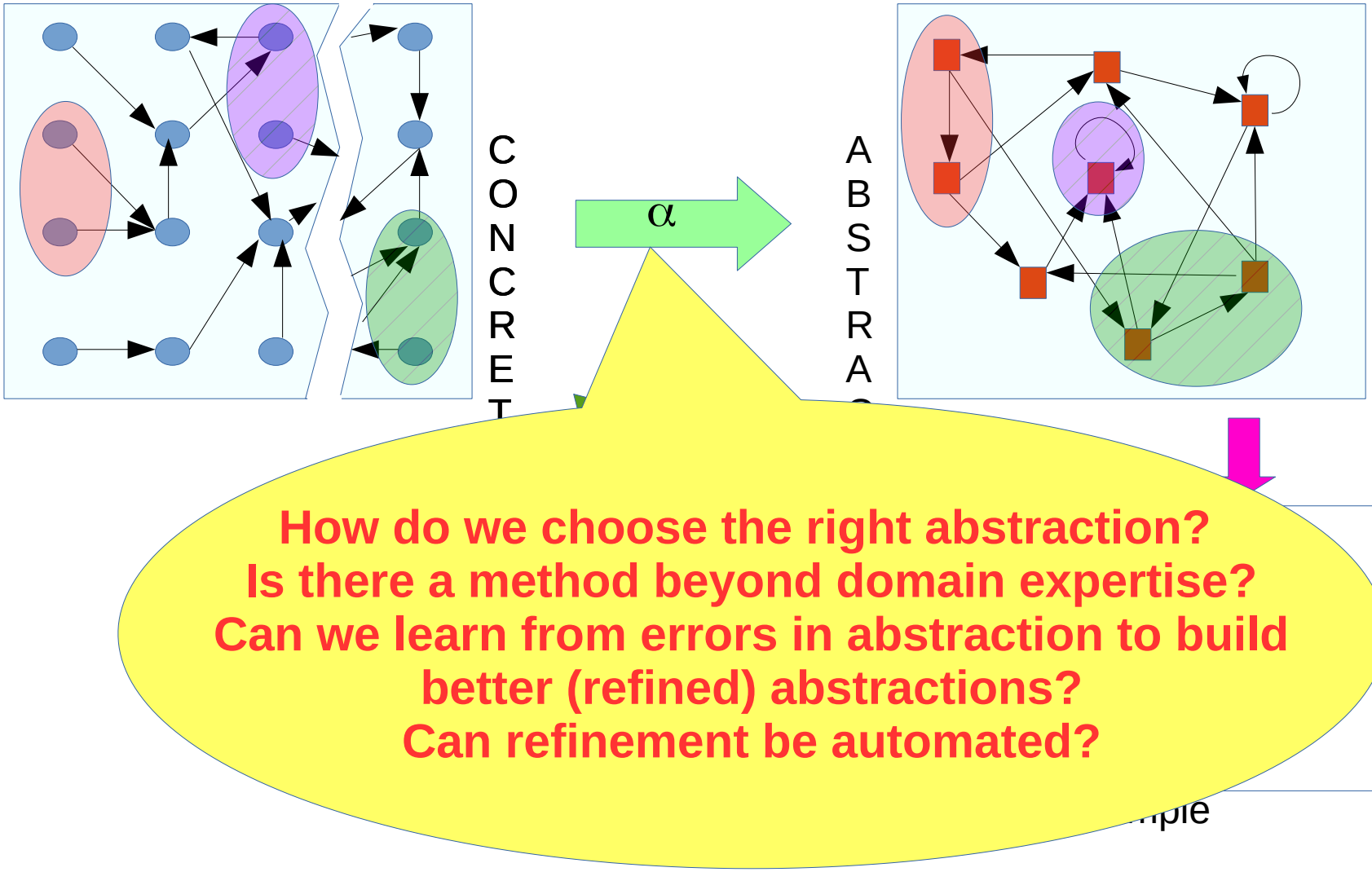
Pre-condition:

NN computation as a sequence of state transitions

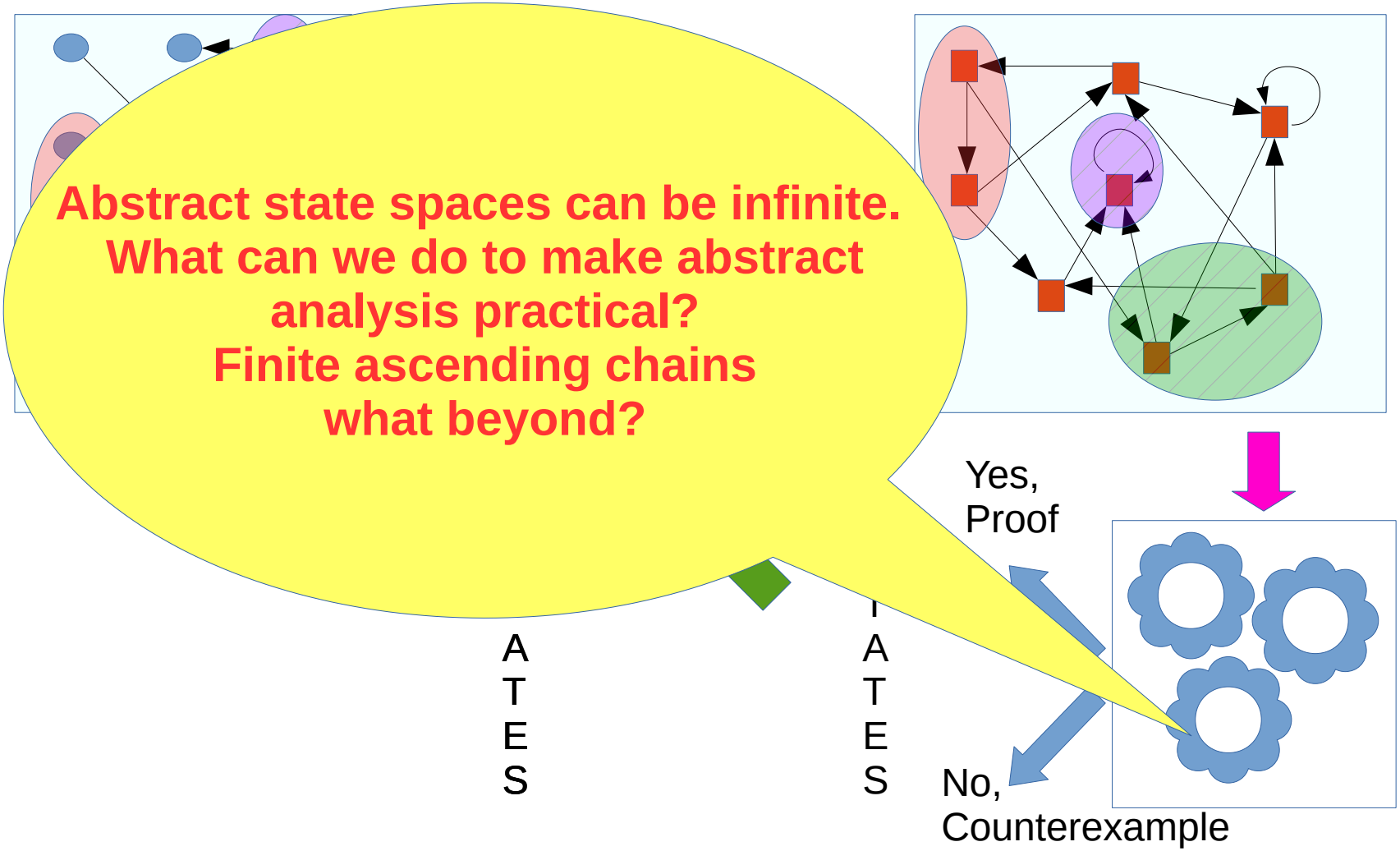
Post-condition:

Abstract analysis engine

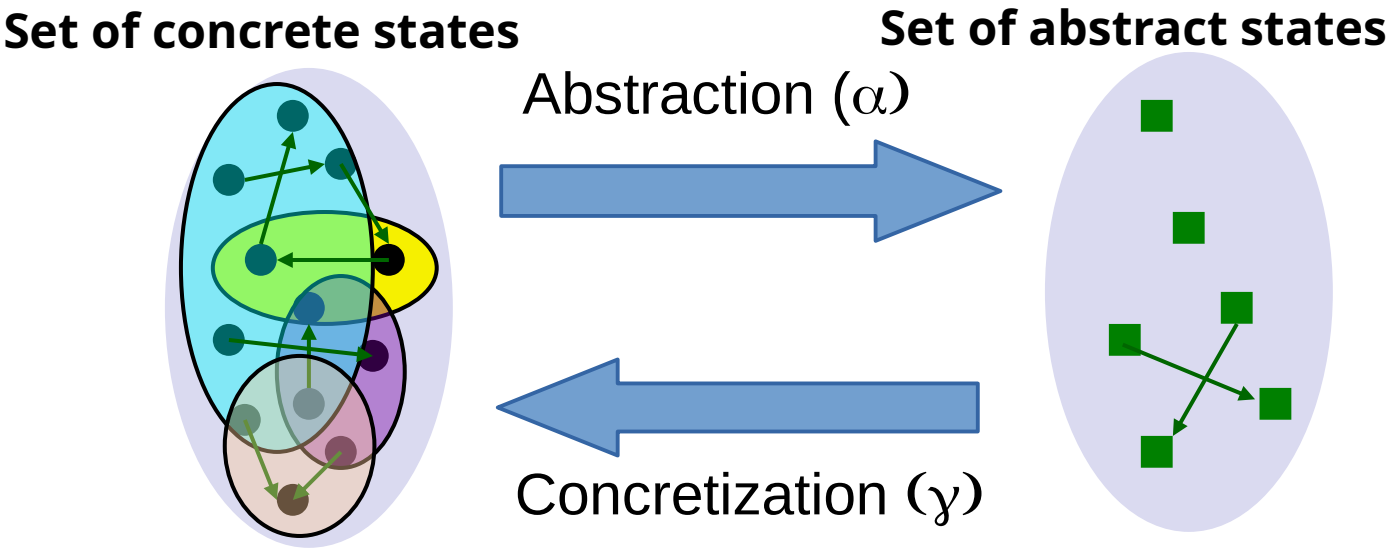
The Game Plan



The Game Plan



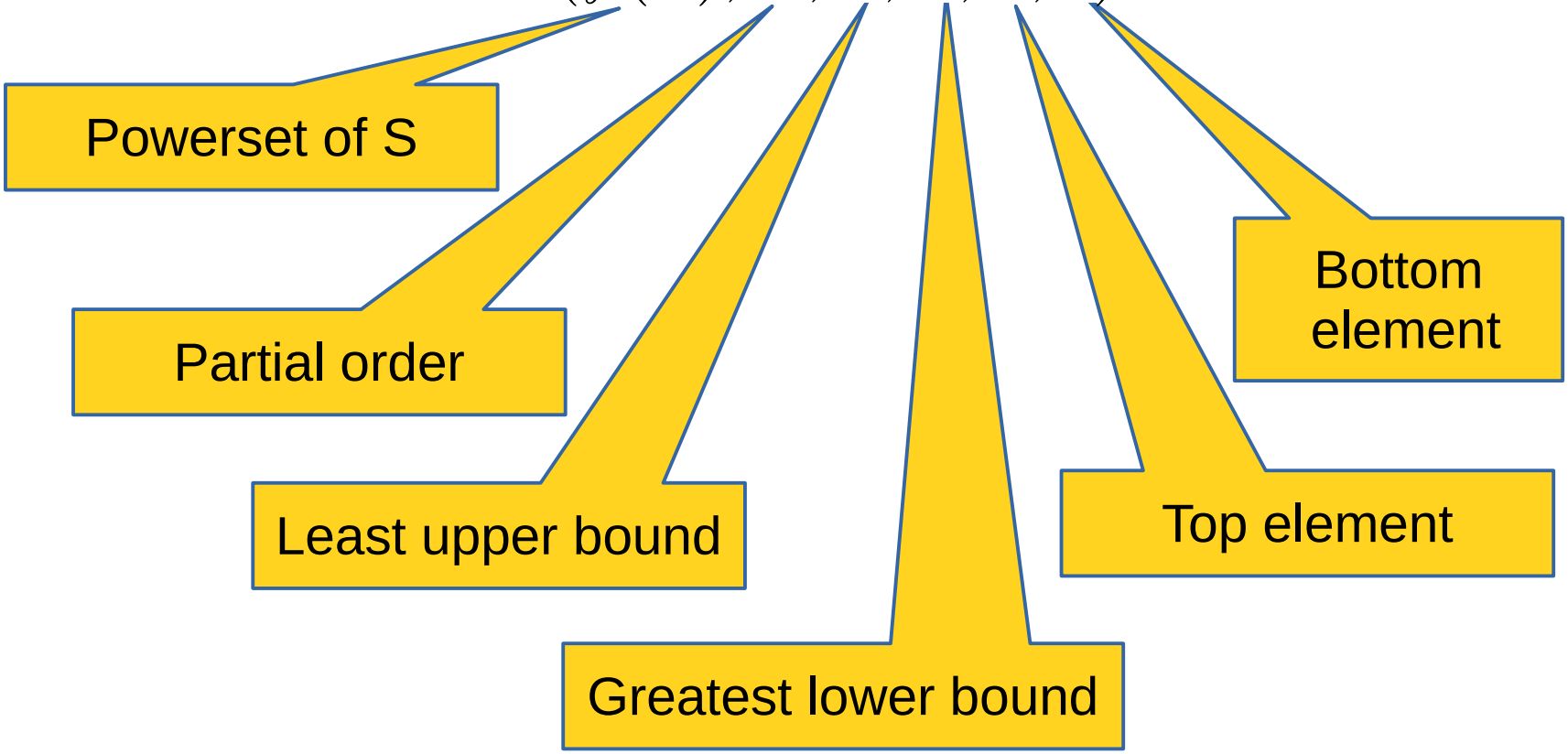
Desirable Properties of Abstraction



- Suppose $S_1 \subseteq S_2$: subsets of concrete states
  - Any behaviour starting from a state S_1 can also happen starting from some state in S_2
  - If $S_1 \subseteq S_2$, we want this monotonicity in behaviour in abstr state space too
    - Need ordering of abstract states, similar in spirit to $\subseteq$

Structure of Concrete State Space

- Set of concrete states: S
  - Concrete lattice C = $(\wp(S), \subseteq, \cup, \cap, S, \emptyset)$



Structure of Abstract State Space

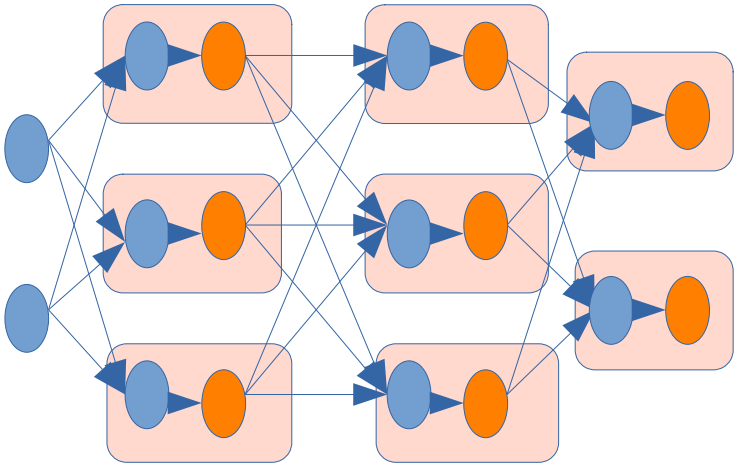
- Abstract lattice A = $(\mathcal{A}, \sqsubseteq, \sqcup, \sqcap, \top, \bot)$
- Abstraction function $\alpha : \wp(S) \to \mathcal{A}$
  - Monotone: $S_1 \subseteq S_2 \Rightarrow \alpha(S_1) \sqsubseteq \alpha(S_2)$ $S_1, S_2 \subseteq S$
  - $\alpha(S) = \top, \quad \alpha(\emptyset) = \bot$
- Concretization function $\gamma : \mathcal{A} \to \wp(S)$
  - Monotone: $a_1 \sqsubseteq a_2 \Rightarrow \gamma(a_1) \subseteq \gamma(a_2)$ $a_1, a_2 \in \mathcal{A}$
  - $\gamma(\top) = S, \quad \gamma(\bot) = \emptyset$

A Simple Abstract Domain

Interval Abstract Domain

- Simplest domain for analyzing numerical programs
- Represent values of each variable separately using intervals
- Example:

Pre-condition on (x1, x2)



Post-condition on (x17, x18)

Represent values of inputs by intervals,

Compute values of hidden layer nodes and outputs as intervals

Interval Abstract Domain

- Abstract states: intervals of values of x, (ignore values of y)
  - [-10, 7]: { (x, y) | -10 <= x <= 7 }
  - (-∞, 20]: { (x, y) | x <= 20 }

- $\sqsubseteq$ relation: Inclusion of intervals
  [-10, 7] $\sqsubseteq$ [-20, 9]
- $\sqcup$ and $\sqcap$: union and intersection of intervals
  [-10, 9] $\sqcup$ [-20, 7] = [-20, 9]
  [-10, 9] $\sqcap$ [-20, 7] = [-10, 7]
- $\bot$ is empty interval of x
- $\top$ is (-∞, +∞)

Interval Abstract Domain

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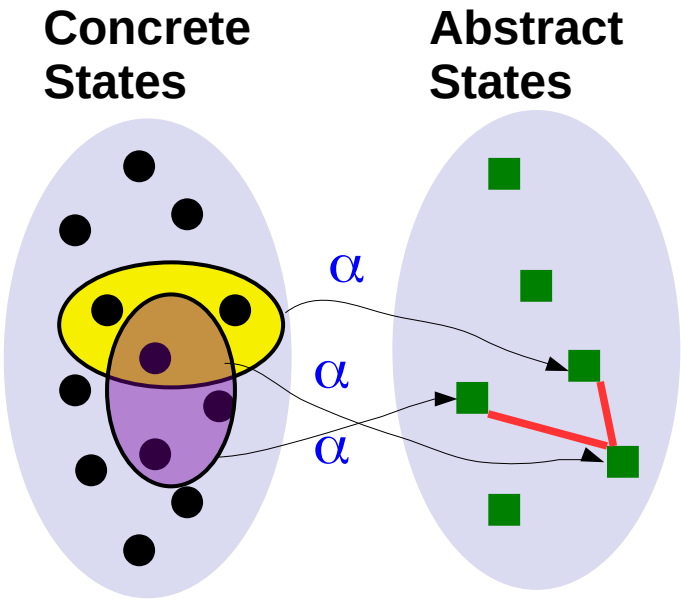

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  [-10, 7] $\sqsubseteq$ [-20, 9]
- $\sqcup$ and $\sqcap$: union and intersection
  [-10, 9] $\sqcup$ [-20, 7] = [-20, 9]
  [-10, 9] $\sqcap$ [-20, 7] = [-10, 7]
- $\bot$ is empty interval of x
- $\top$ is (-∞, +∞)

α({(1, 3), (2, 4), (5, 7)}) = [1, 5]
α({(5, 7), (7, 6), (9, 10)}) = [5, 9]
α({(5, 7)}) = [5, 5]

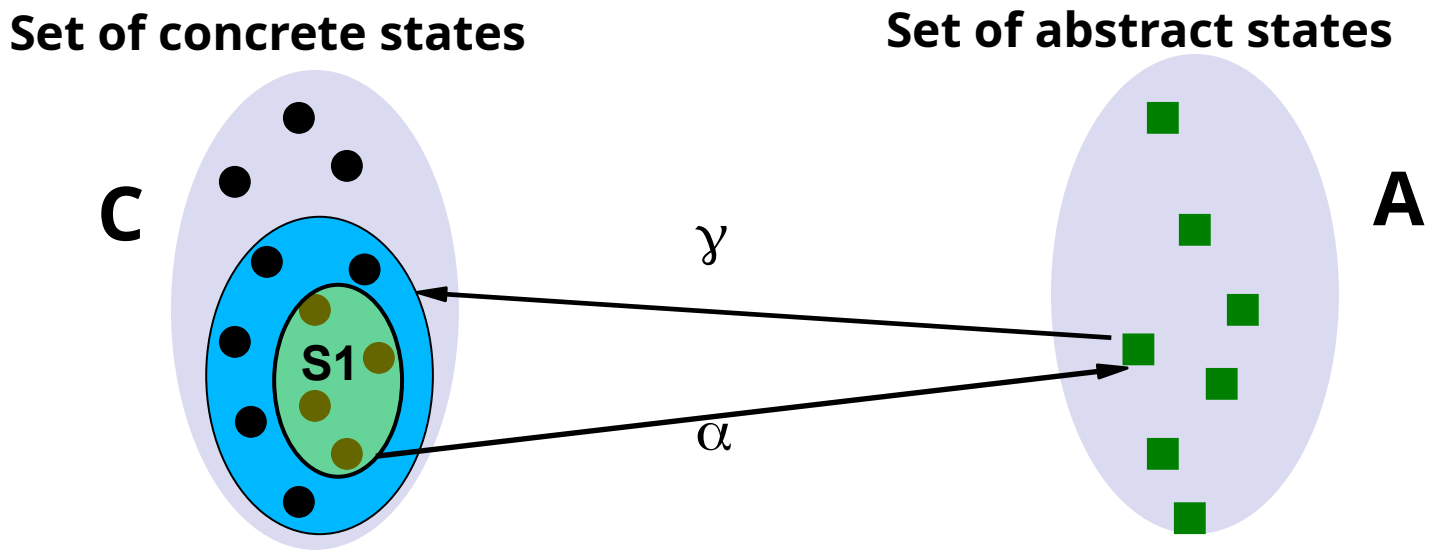
Interval Abstract Domain

- Abstract states: pairs of intervals (one for x, y)
  - ([-10, 7] , (-1, 20])
  - $\sqsubseteq$ relation: Inclusion of intervals

    ([-10, 7] , (-1, 20]) $\sqsubseteq$ ([-20, 9], (-1, +∞))
  - $\sqcup$ and $\sqcap$: union and intersection of intervals
  - ([-10, 9] , (-1, 20]) $\sqcap$([-20, 7], [3,+∞)) = ([-10, 7], [3, 20])
    - ([-10, 9], (-1, 20]) $\sqcup$([-20, 7], [3,+∞)) = ([-20, 9],(-1,+∞))
  - $\bot$ is empty interval of x and y
  - $\top$ is ((-∞, +∞), (-∞, +∞))

Desirable Properties of α and γ

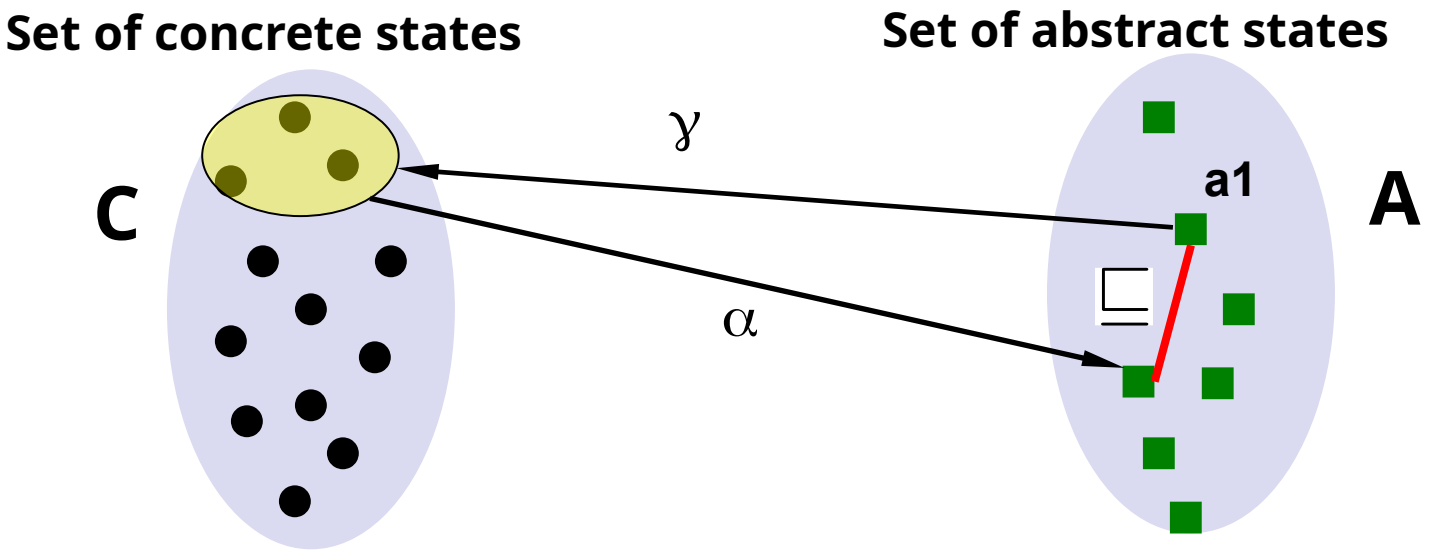
For all $S_1 \subseteq C$ $\quad S_1 \subseteq \gamma(\alpha(S_1))$



Desirable Properties of α and γ

$$S_1 \subseteq \gamma(\alpha(S_1)) \quad \text{forall} \quad S_1 \subseteq \mathcal{C}$$

$$\alpha(\gamma(a_1)) \sqsubseteq a_1 \quad \text{forall} \quad a_1 \in \mathcal{A}$$

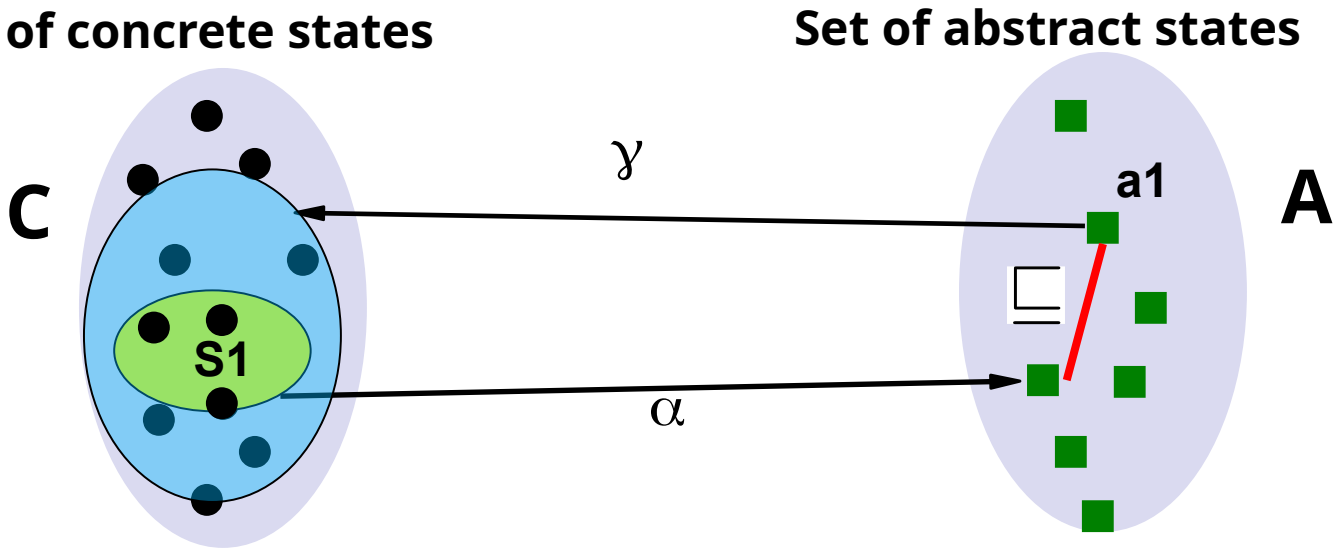


α and γ form a Galois connection

Desirable Properties of α and γ

- α and γ form a Galois connection
  - Second (equivalent) view:

$$\alpha(S_1) \sqsubseteq a_1 \Leftrightarrow S_1 \subseteq \gamma(a_1) \text{ for all } S_1 \subseteq S,\ a_1 \in \mathcal{A}$$



Homework Problem

Let $\alpha : (S) \to \mathcal{A}$ and $\gamma : \mathcal{A} \to (S)$ be monotone maps

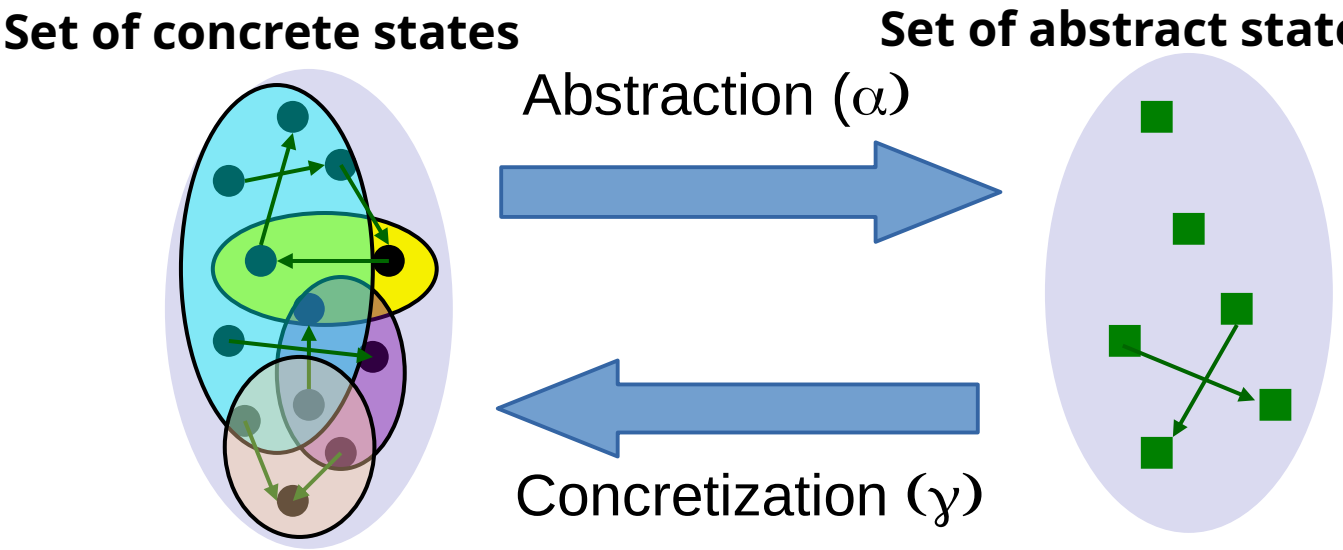
Show that

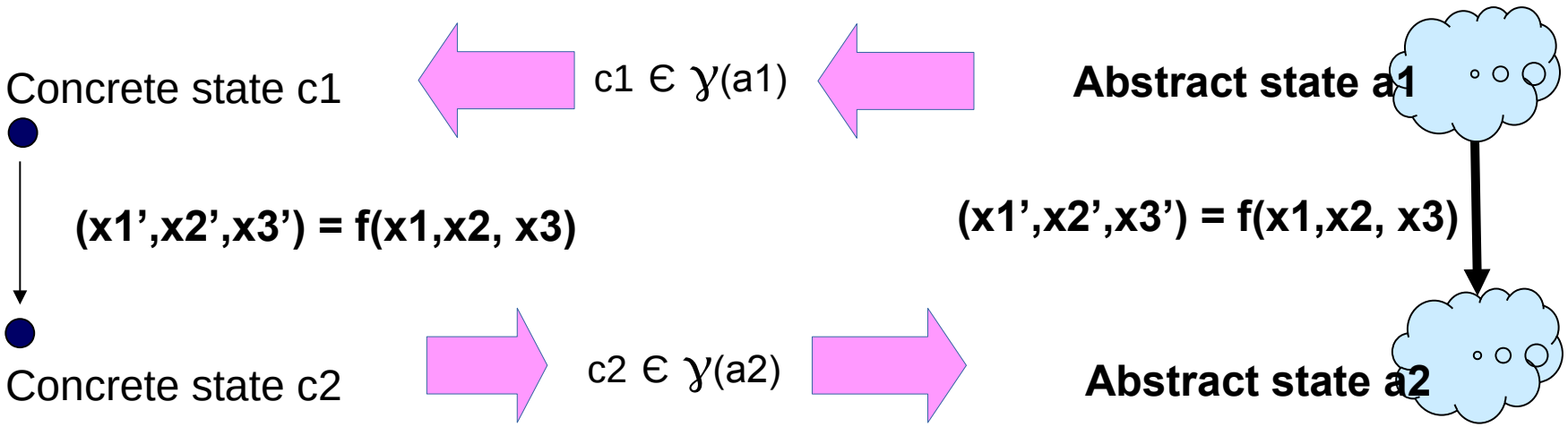
$S_1 \subseteq \gamma(\alpha(S_1))$ for all $S_1 \in (S)$ and $\alpha(\gamma(a_1)) \sqsubseteq a_1$ for all $a_1 \in \mathcal{A}$

holds if and only if

$\alpha(S_1) \sqsubseteq a_1 \Leftrightarrow S_1 \subseteq \gamma(a_1)$ for all $S_1 \in (S)$ and $a_1 \in \mathcal{A}$

Computing Abstract State Transitions

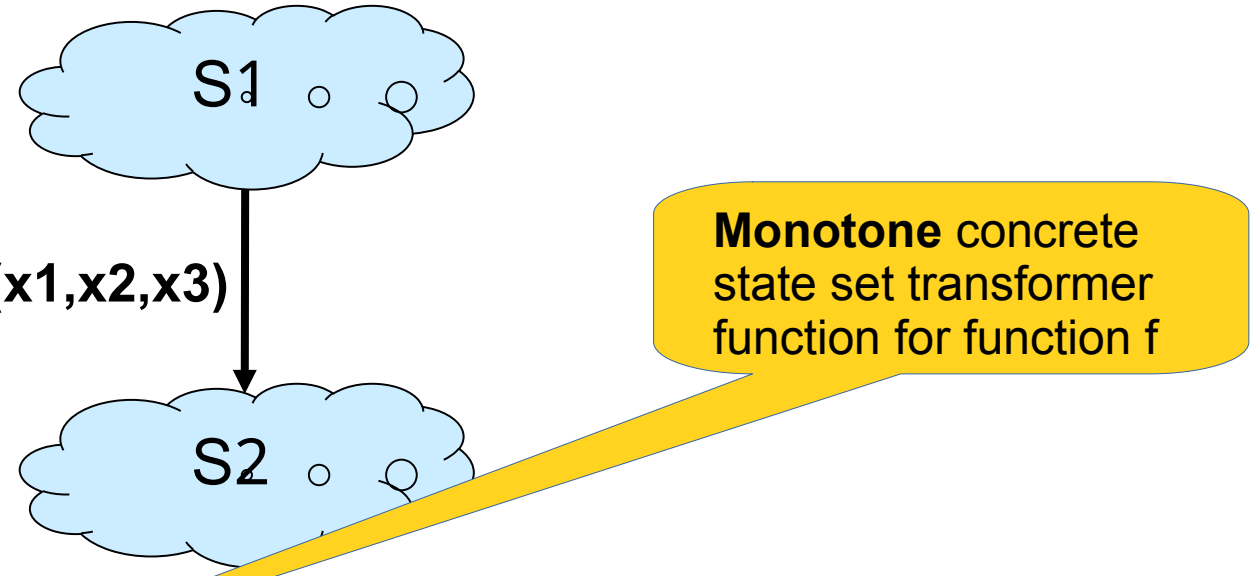




Computing Abstract State Transitions

- Concrete state set transformer function
  - Example:

S1 = { (x1, x2, x3) | ….. }: set of concr. states

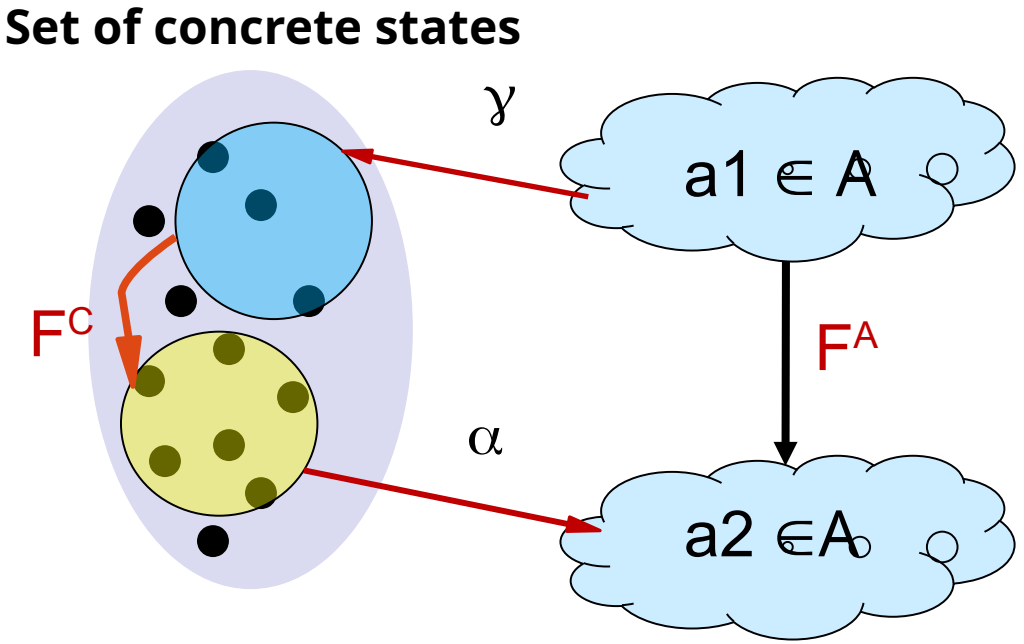


S2 = {(x1', x2', x3') | ∃ (x1, x2, x3) ∈ S1, (x1',x2',x3') = f(x1,x2, x3)}

= F^C(S1) : set of concrete states

Computing Abstract State Transitions

- Abstract state transformer function
  - Example:



a2 = α(F^C (γ (a1))) ideally, but F^A(a1) $\sqsupseteq$ α(F^C (γ (a1))) often used

Summary

- Abstract interpretation is a general framework for analysis of state transition systems
- Widely used for verification and static analysis of programs
- Recent applications in neural network analysis
- Choice of right abstraction crucial to success
  - Balance between precision and efficiency