Fun using Big-M Encoding in MILP

Supratik Chakraborty

Conditional Linear Constraints

b	integer variable, takes values in $\{0, 1\}$
$\mathbf{A_1}, \mathbf{A_2}$	$n \times m$ real matrices
$\mathbf{x}$	$m \times 1$ vector of real-valued variables
$\mathbf{d_1}, \mathbf{d_2}$	$n \times 1$ vector of reals

$$\text{Encode} \quad \begin{array}{lcl} (b = 1) & \Rightarrow & \mathbf{A_1 x} \leq \mathbf{d_1} \qquad \wedge \\ (b = 0) & \Rightarrow & \mathbf{A_2 x} \leq \mathbf{d_2} \end{array}$$

as a Mixed Integer Linear (MIL) constraint

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as a Mixed Integer Linear (MIL) constraint

Let $M \geq (\mathbf{A_i x} - \mathbf{d_i})_\mathbf{j}$ for $1 \leq i \leq 2$ and $1 \leq j \leq n$

Let $\mathbf{u}$ be the all-1s vector of dimension $n \times 1$

$$\mathbf{A_1 x} \leq \mathbf{d_1} + M.(1 - b).\mathbf{u} \quad \wedge$$
$$\mathbf{A_2 x} \leq \mathbf{d_2} + M.b.\mathbf{u}$$

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Finding smallest M s.t. $M \geq (\mathbf{A_i x} - \mathbf{d_i})_\mathbf{j}$ for $1 \leq i \leq 2$, $1 \leq j \leq n$?

For $1 \leq j \leq n$, use **Linear Programming** to find

$$m_{1,j} = \max\left(\mathbf{A_1 x} - \mathbf{d_1}\right)_\mathbf{j} \text{ subject to } \mathbf{A_2 x} \leq \mathbf{d_2}$$

$$m_{2,j} = \max\left(\mathbf{A_2 x} - \mathbf{d_2}\right)_\mathbf{j} \text{ subject to } \mathbf{A_1 x} \leq \mathbf{d_1}$$

Then choose $M \geq \max\left(0, \max_{i=1}^{2} \max_{j=1}^{n} m_{i,j}\right)$

Disjunction of Linear Constraints

$\mathbf{A_1}, \mathbf{A_2}$	$n \times m$ real matrices
$\mathbf{x}$	$m \times 1$ vector of real-valued variables
$\mathbf{d_1}, \mathbf{d_2}$	$n \times 1$ vector of reals

Encode $\mathbf{A_1}\mathbf{x} \leq \mathbf{d_1} \quad \vee \quad \mathbf{A_2}\mathbf{x} \leq \mathbf{d_2}$

as a Mixed Integer Linear (MIL) constraint

Disjunction of Linear Constraints

$\mathbf{A_1}, \mathbf{A_2}$	$n \times m$ real matrices
$\mathbf{x}$	$m \times 1$ vector of real-valued variables
$\mathbf{d_1}, \mathbf{d_2}$	$n \times 1$ vector of reals

Encode $\mathbf{A_1 x} \leq \mathbf{d_1} \quad \vee \quad \mathbf{A_2 x} \leq \mathbf{d_2}$ as a MIL constraint

Let $M \geq (\mathbf{A_i x} - \mathbf{d_i})_\mathbf{j}$ for $1 \leq i \leq 2$ and $1 \leq j \leq n$

Let $\mathbf{u}$ be the all-1s vector of dimension $n \times 1$

Let b be an integer variable

$$\mathbf{A_1 x} \leq \mathbf{d_1} + M.(1-b).\mathbf{u} \quad \wedge$$
$$\mathbf{A_2 x} \leq \mathbf{d_2} + M.b.\mathbf{u} \quad \wedge$$
$$0 \leq b \leq 1$$

Multiple Disjoined Linear Constraints

$\mathbf{A_1}, \mathbf{A_2}, \ldots \mathbf{A_k}$	$n \times m$ real matrices
$\mathbf{x}$	$m \times 1$ vector of real-valued variables
$\mathbf{d_1},\, \mathbf{d_2},\, \ldots \mathbf{d_k}$	$n \times 1$ real vectors

Encode $\mathbf{A_1 x} \leq \mathbf{d_1} \quad \vee \quad \mathbf{A_2 x} \leq \mathbf{d_2} \cdots \quad \vee \ \mathbf{A_k x} \leq \mathbf{d_k}$ as a MIL constraint

Multiple Disjoined Linear Constraints

Let $M \geq (\mathbf{A_i x} - \mathbf{d_i})_{\mathbf{j}}$ for $1 \leq i \leq k$ and $1 \leq j \leq n$

Let $\mathbf{u}$ be the all-1s vector of dimension $n \times 1$

Let $b_1, \ldots b_k$ be integer variables

$$\mathbf{A_1 x} \leq \mathbf{d_1} + M.(1 - b_1).\mathbf{u} \quad \wedge \; 0 \leq b_1 \leq 1 \wedge$$

$$\vdots$$

$$\mathbf{A_k x} \leq \mathbf{d_k} + M.(1 - b_k).\mathbf{u} \quad \wedge \; 0 \leq b_k \leq 1 \wedge$$

$$b_1 + \cdots b_k = 1$$

Easy Exercise

Let $x_1, \ldots x_k, z$ be real variables

Encode $z = \max(x_1, \ldots x_k)$ and $z = \min(x_1, \ldots x_k)$
as MIL constraints using big-M encoding

Hint: Recall $z = \max(x_1, \ldots x_k)$ is equivalent to
$\bigwedge_{i=1}^{k}(z \geq x_i) \;\wedge\; \bigvee_{i=1}^{k}(z \leq x_i)$

Analogously, for $z = \min(x_1, \ldots x_k)$