CS781:
A Quick Primer on Abstract Interpretation for Neural Networks

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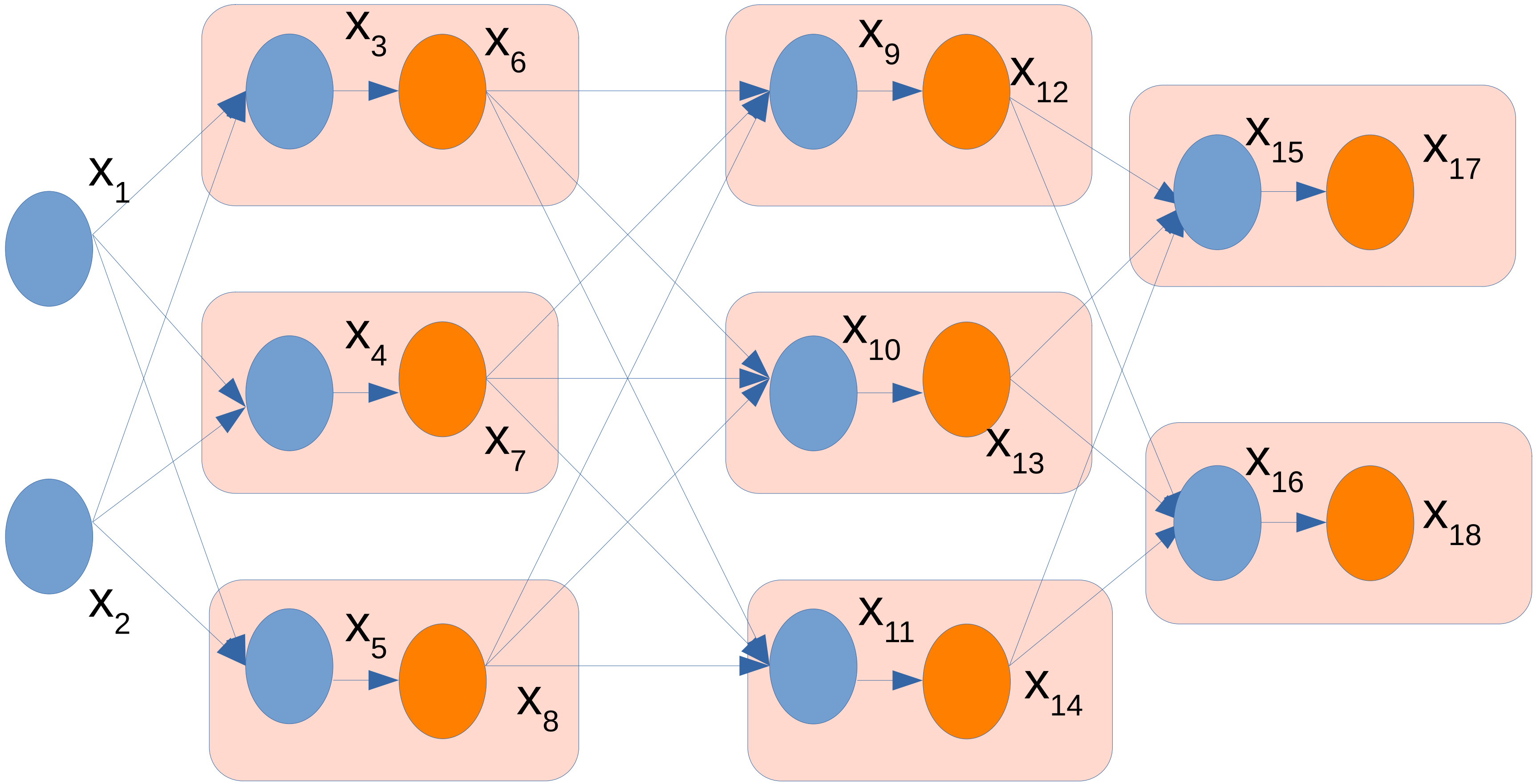
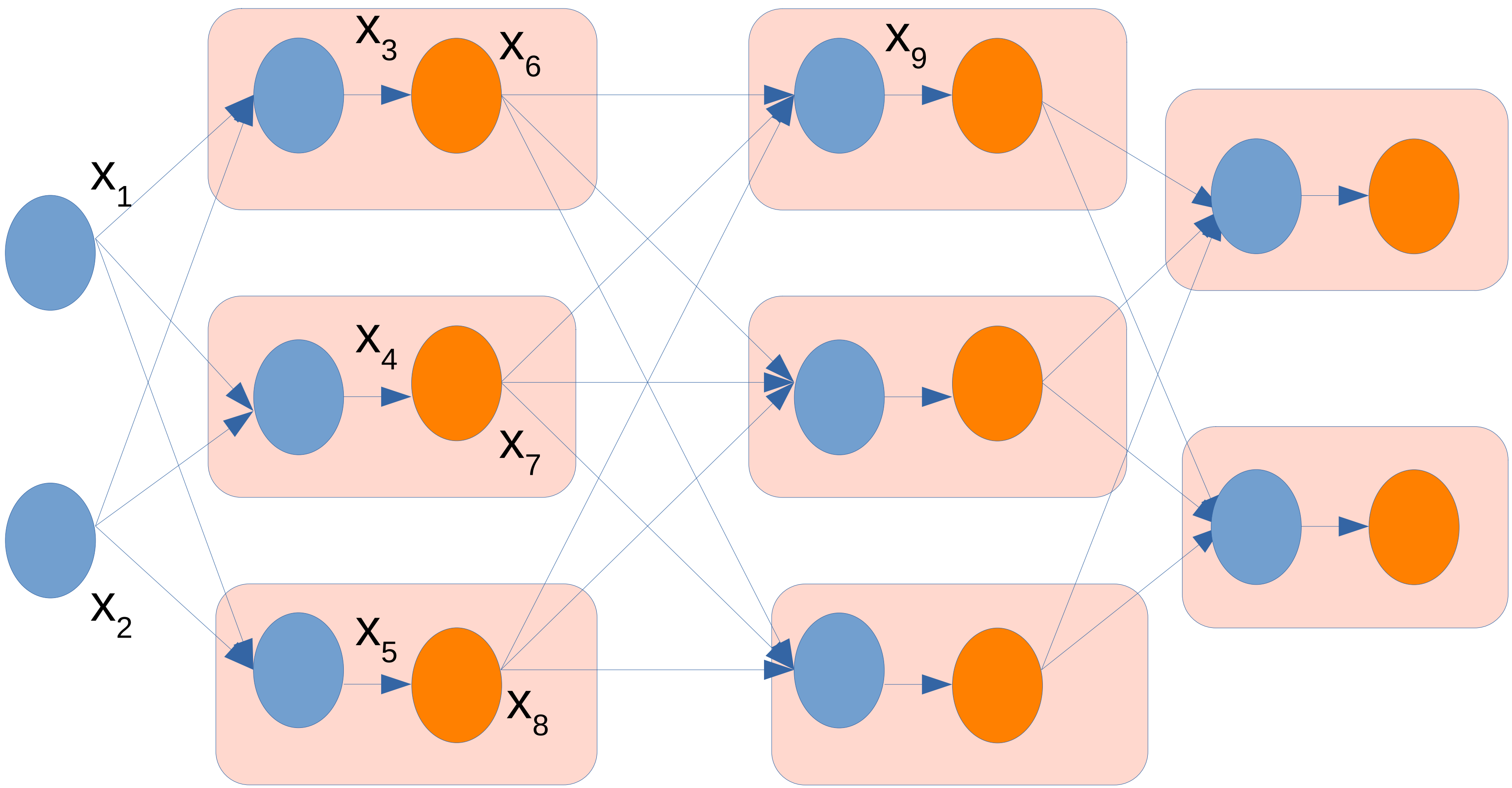
Notion of State in Neural Network



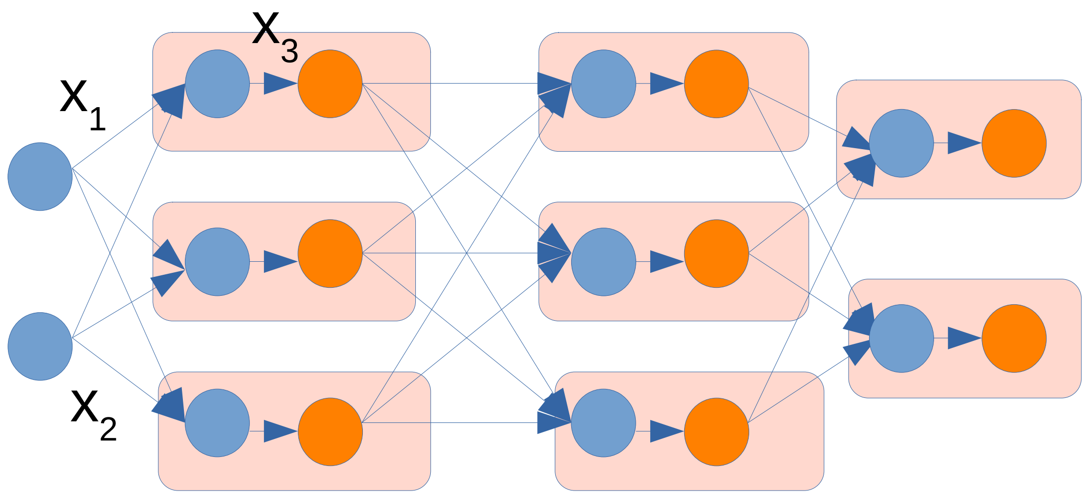
State: $(x_1, x_2, \ldots x_{18})$ in R^{18}

State Change in Feed-Forward Neural Network



$(x'_1, x'_2, \ldots x'_{i-1,} x'_i) = f_i(x_1, x_2, \ldots x_{i-1}),$ for i in {3, ..., 18 }

State Change in Feed-Forward NN as a sequence of instrns



$(x'_1, x'_2, x'_3) = f_3(x_1, x_2);$

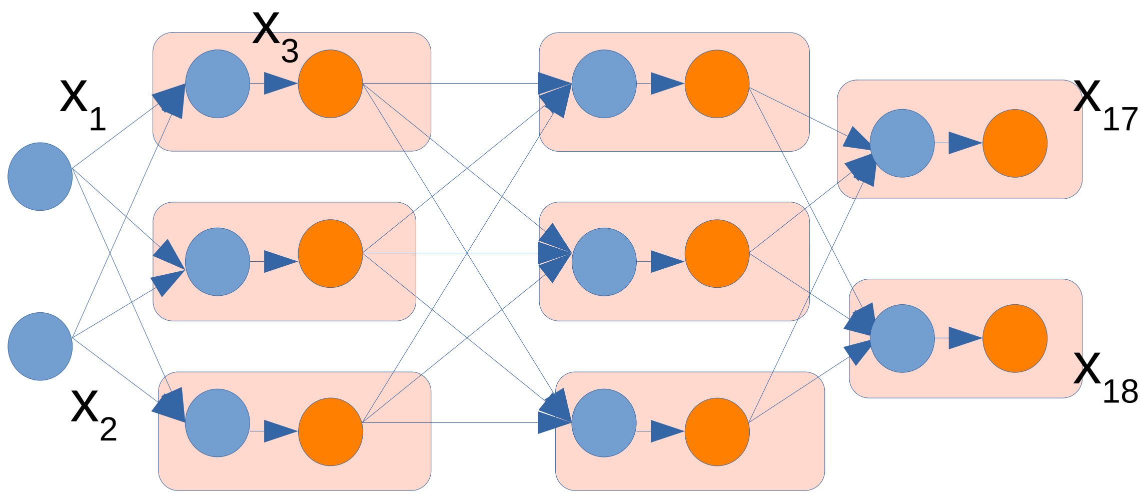
$(x''_1, x''_2, x''_3, x''_4) = f_4(x'_1, x'_2, x'_3);$

...

NN computation: a sequence of state transitions caused by seq of instructions

Proving Property of a FF NN

Pre-condition on (x1, x2)



Post-condition on (x17, x18)

{Pre-condition on (x1, x2)}

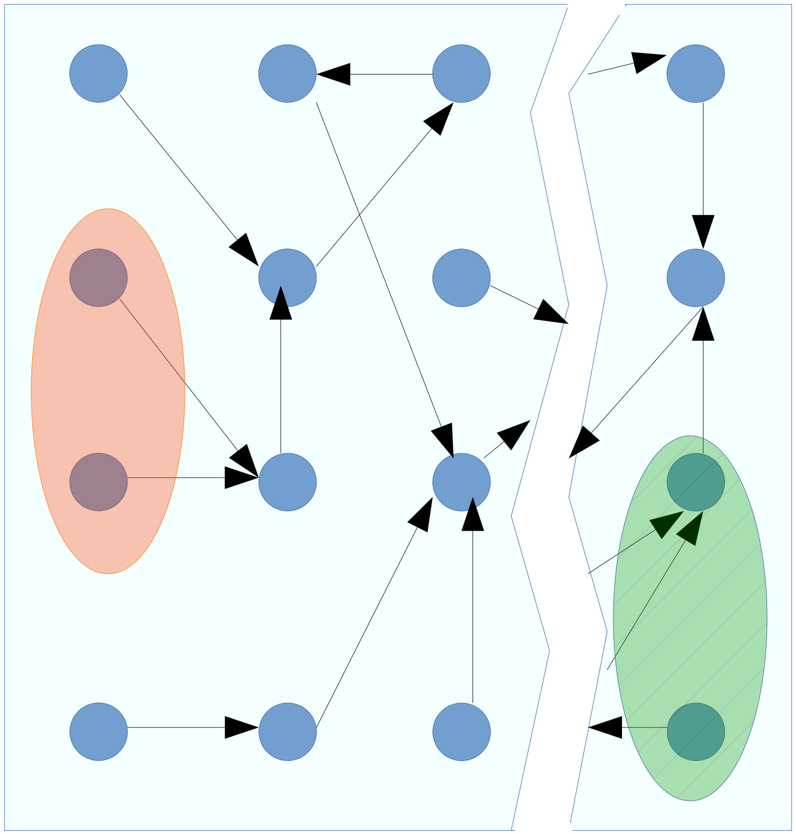
$$(x'_1, x'_2, x'_3) = f_3(x_1, x_2);$$

$$(x''_1, x''_2, x''_3, x''_4) = f_4(x'_1, x'_2, x'_3);$$

...

{Post-condition on (x17, x18) }

NN Computation as a State Transition System



{Pre-condition on (x1, x2)}

$(x'_1, x'_2, x'_3) = f_3(x_1, x_2);$

$(x''_1, x''_2, x''_3, x''_4) = f_4(x'_1, x'_2, x'_3);$

…

{Post-condition on (x17, x18) }

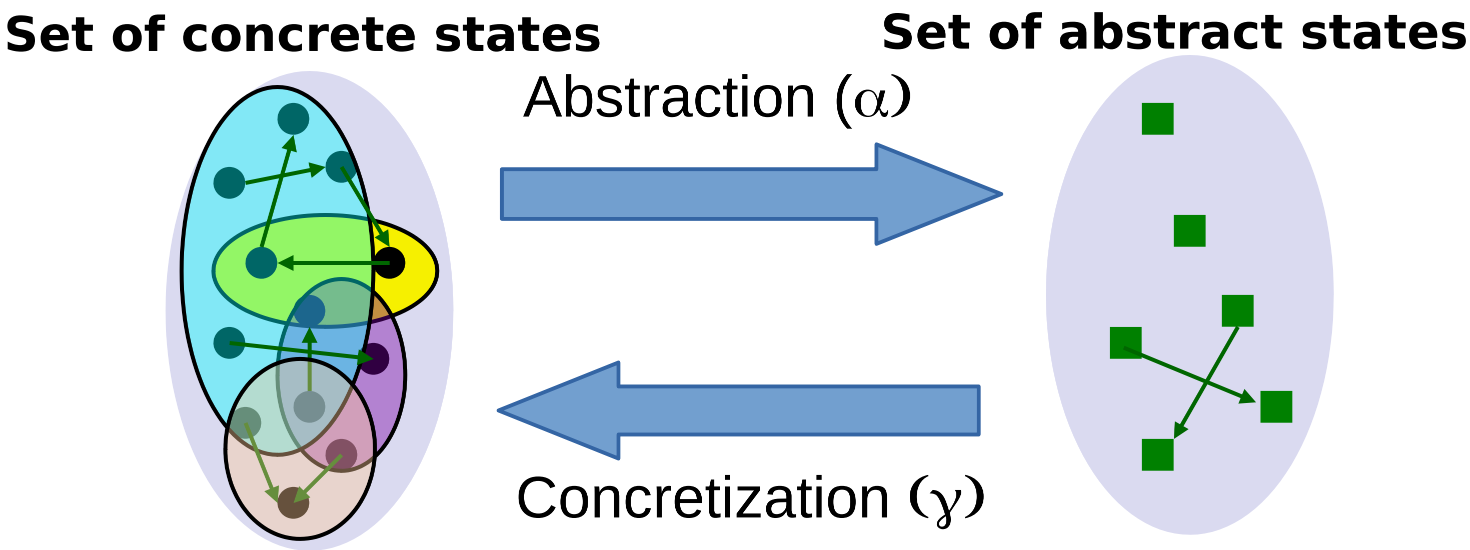
Dealing with State Space Size

- Infinite state space
  - Difficult to represent using state transition diagram
  - Can we still do some reasoning?
- Solution: Use of abstraction
  - Naive view
    - Bunch sets of states together "intelligently"
    - Don't talk of individual states, talk of a representation of a set of states
    - Transitions between state set representations
  - Granularity of reasoning shifted
  - Extremely powerful general technique
    - Allows reasoning about large/infinite state spaces

Concrete states

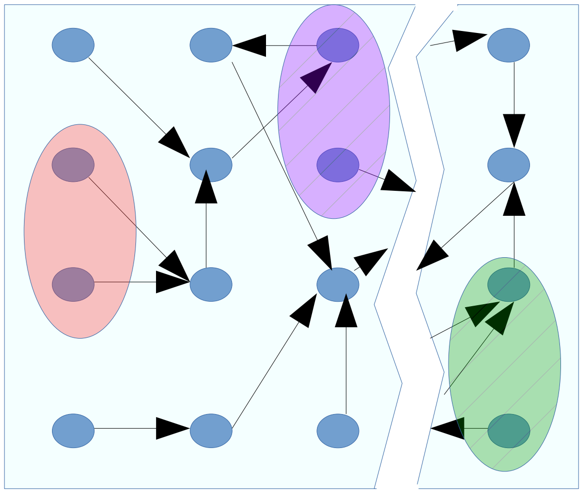
Abstract states

A Generic View of Abstraction

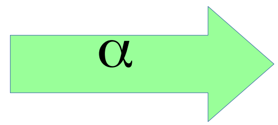


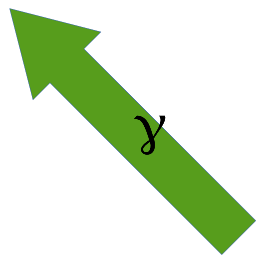
- Every subset of concrete states mapped to unique abstract state
- Desirable to capture containment relations
- Transitions between state sets (abstract states)

The Game Plan

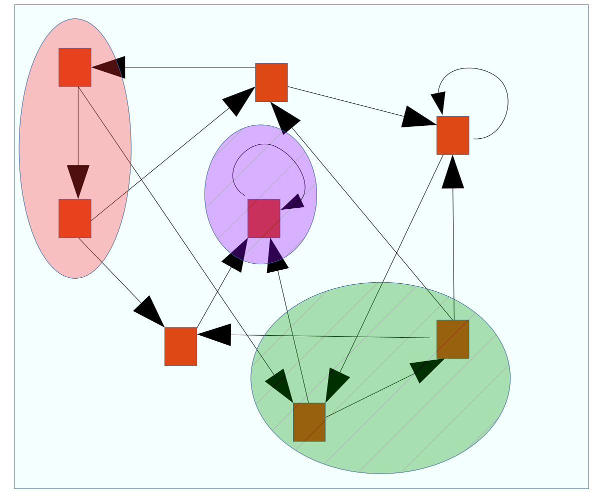


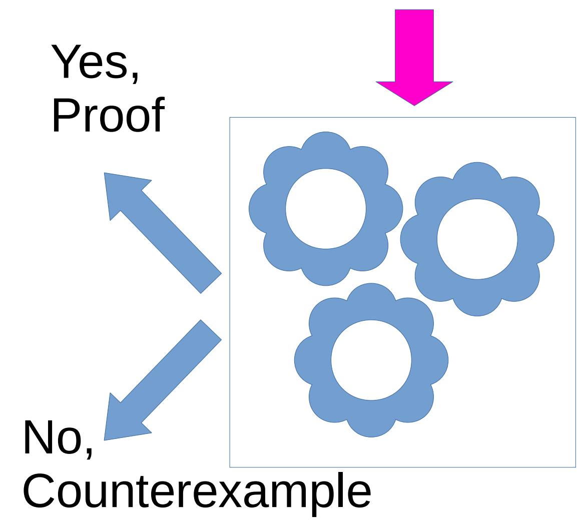
CONCRETE STATES





ABSTRACT STATES





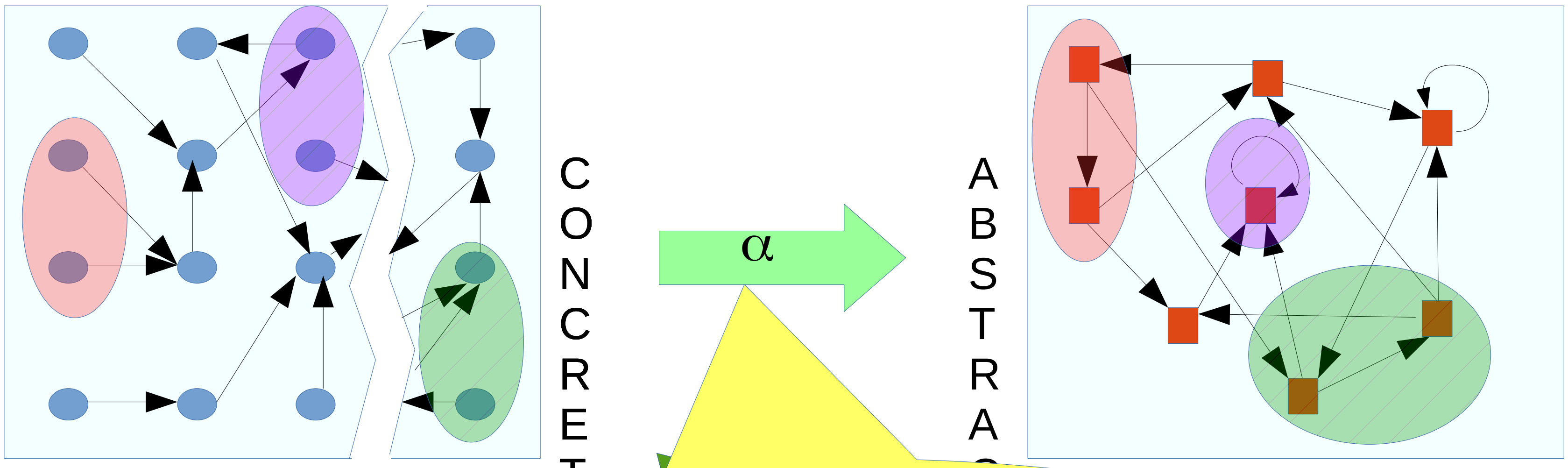
Abstract analysis engine

Pre-condition:

NN computation as a sequence of state transitions

Post-condition:

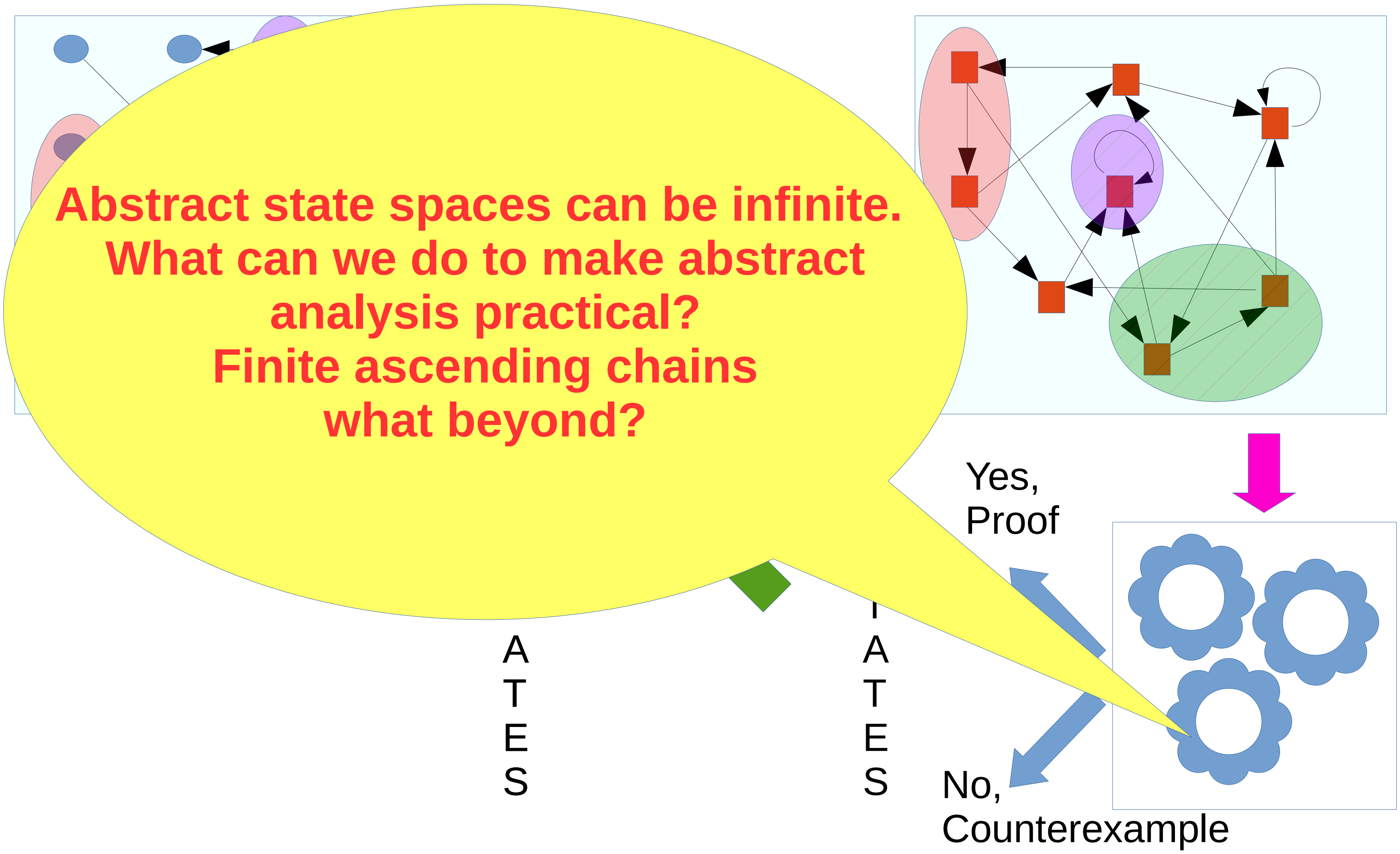
The Game Plan



How do we choose the right abstraction?
Is there a method beyond domain expertise?
Can we learn from errors in abstraction to build better (refined) abstractions?
Can refinement be automated?

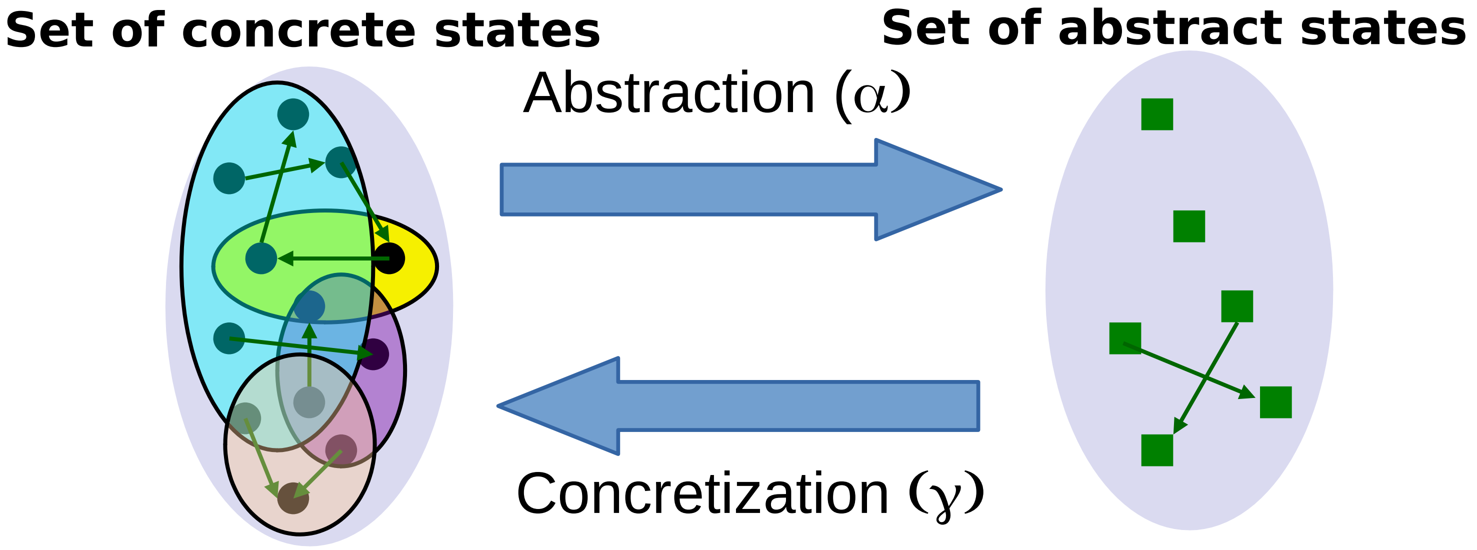
Abstract analysis engine

The Game Plan



Abstract analysis engine

Desirable Properties of Abstraction



- Suppose $S_1 \subseteq S_2$: subsets of concrete states
  - Any behaviour starting from S_1 can also happen starting from S_2
  - If $\alpha(S_1) = a_1, \alpha(S_2) = a_2$ we want this monotonicity in behaviour in abstr state space too
    - Need ordering of abstract states, similar in spirit to $S_1 \subseteq S_2$

Structure of Concrete State Space

- Set of concrete states: S
  - Concrete lattice C $= (\wp(S), \subseteq, \cup, \cap, S, \emptyset)$

Powerset of S

Partial order

Least upper bound

Greatest lower bound

Top element

Bottom element

Structure of Abstract State Space

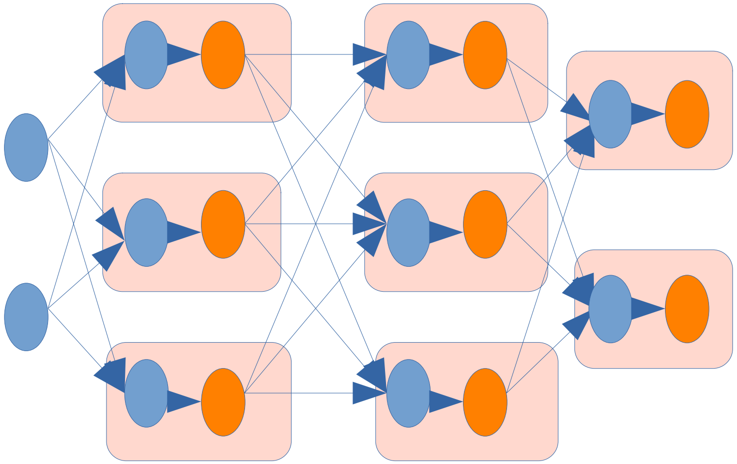
- Abstract lattice $\mathsf{A} = (\mathcal{A}, \sqsubseteq, \sqcup, \sqcap, \top, \bot)$
- Abstraction function $\alpha : \wp(S) \rightarrow \mathcal{A}$
  - Monotone: $S_1 \subseteq S_2 \Rightarrow \alpha(S_1) \sqsubseteq \alpha(S_2)$ for all $S_1, S_2 \subseteq S$
  - $\alpha(S) = \top, \quad \alpha(\emptyset) = \bot$
- Concretization function $\gamma : \mathcal{A} \rightarrow \wp(S)$
  - Monotone: $a_1 \sqsubseteq a_2 \Rightarrow \gamma(a_1) \subseteq \gamma(a_2)$ for all $a_1, a_2 \in \mathcal{A}$
  - $\gamma(\top) = S, \quad \gamma(\bot) = \emptyset$

A Simple Abstract Domain

Interval Abstract Domain

- Simplest domain for analyzing numerical programs
- Represent values of each variable separately using intervals
- Example:

Pre-condition on (x1, x2)



Post-condition on (x17, x18)

Represent values of inputs by intervals,
Compute values of hidden layer nodes and outputs as intervals

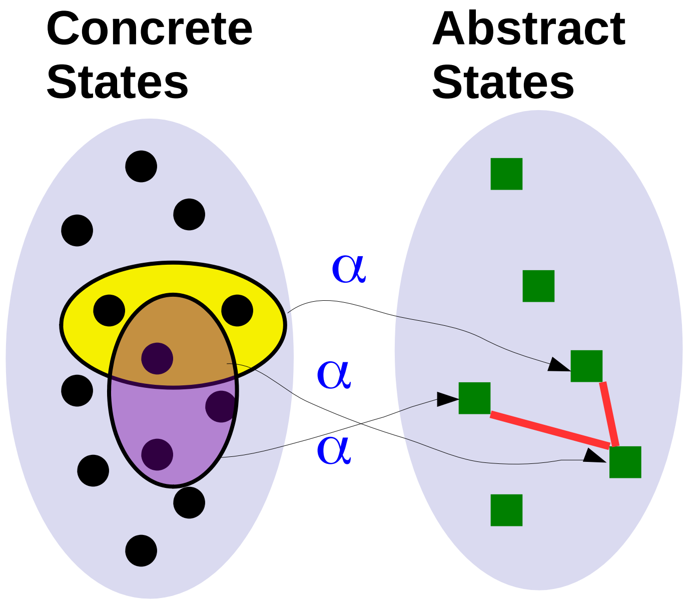
Interval Abstract Domain

- Abstract states: intervals of values of x, (ignore values of y)
    - [-10, 7]: { (x, y) | -10 <= x <= 7 }
    - (-∞, 20]: { (x, y) | x <= 20 }

- $\sqsubseteq$ relation: Inclusion of intervals
    - [-10, 7] $\sqsubseteq$ [-20, 9]
- $\sqcup$ and $\sqcap$: union and intersection of intervals
    - [-10, 9] $\sqcup$ [-20, 7] = [-20, 9]
    - [-10, 9] $\sqcap$ [-20, 7] = [-10, 7]
- $\bot$ is empty interval of x
- $\top$ is (-∞, +∞)

Interval Abstract Domain

- Abstract states: intervals of values of x, (ignore values of y)
  - [-10, 7]: { (x, y) | -10 <= x <= 7 }
  - (-∞, 20]: { (x, y) | x <= 20 }
  - ⊑ relation: Inclusion of intervals
    - [-10, 7] ⊑ [-20, 9]
  - ⊔ and ⊓ : union and intersection
    - [-10, 9] ⊔ [-20, 7] = [-20, 9]
    - [-10, 9] ⊓ [-20, 7] = [-10, 7]
  - ⊥ is empty interval of x
  - ⊤ is (-∞, +∞)



α({(1, 3), (2, 4), (5, 7)}) = [1, 5]

α({(5, 7), (7, 6), (9, 10)}) = [5, 9]

α({(5, 7)}) = [5, 5]

Interval Abstract Domain

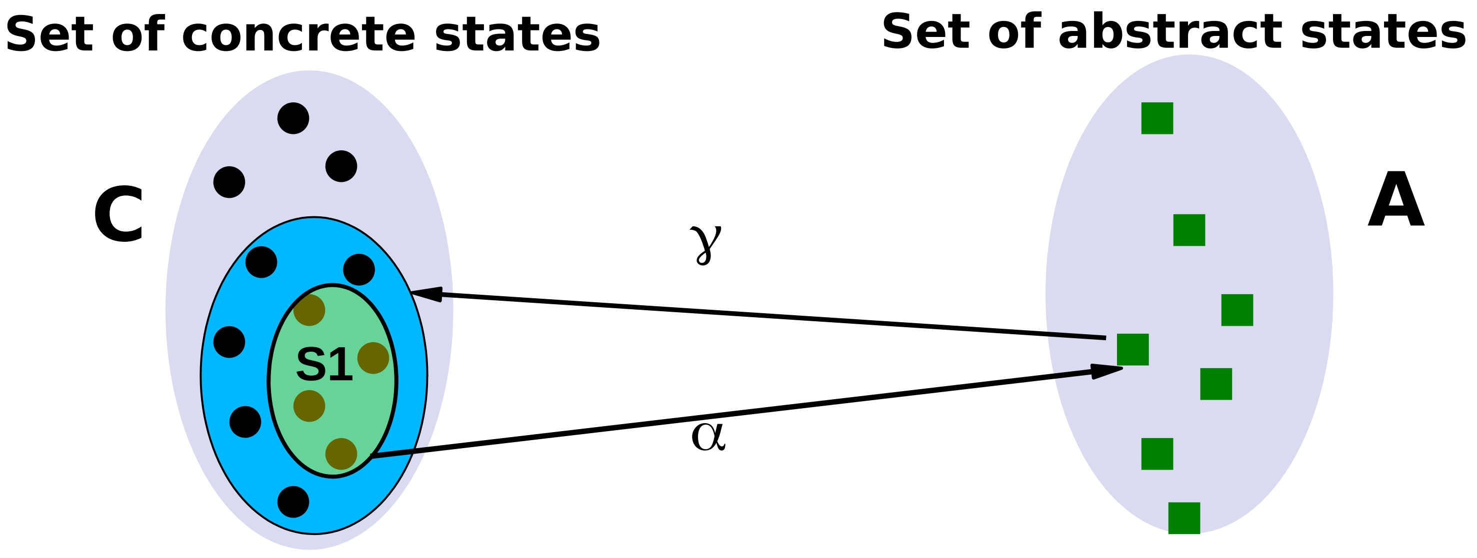
- Abstract states: pairs of intervals (one for x, y)
  - ([-10, 7] , (-1, 20])
  - $\sqsubseteq$ relation: Inclusion of intervals

    ([-10, 7] , (-1, 20]) $\sqsubseteq$ ([-20, 9], (-1, +∞))
  - $\sqcup$ and $\sqcap$: union and intersection of intervals
  - ([-10, 9] , (-1, 20]) $\sqcap$ ([-20, 7], [3,+∞)) = ([-10, 7], [3, 20])
  - ([-10, 9], (-1, 20]) $\sqcup$ ([-20, 7], [3,+∞)) = ([-20, 9],(-1,+∞))

  - $\bot$ is empty interval of x and y
  - $\top$ is ((-∞, +∞), (-∞, +∞))

Desirable Properties of α and γ

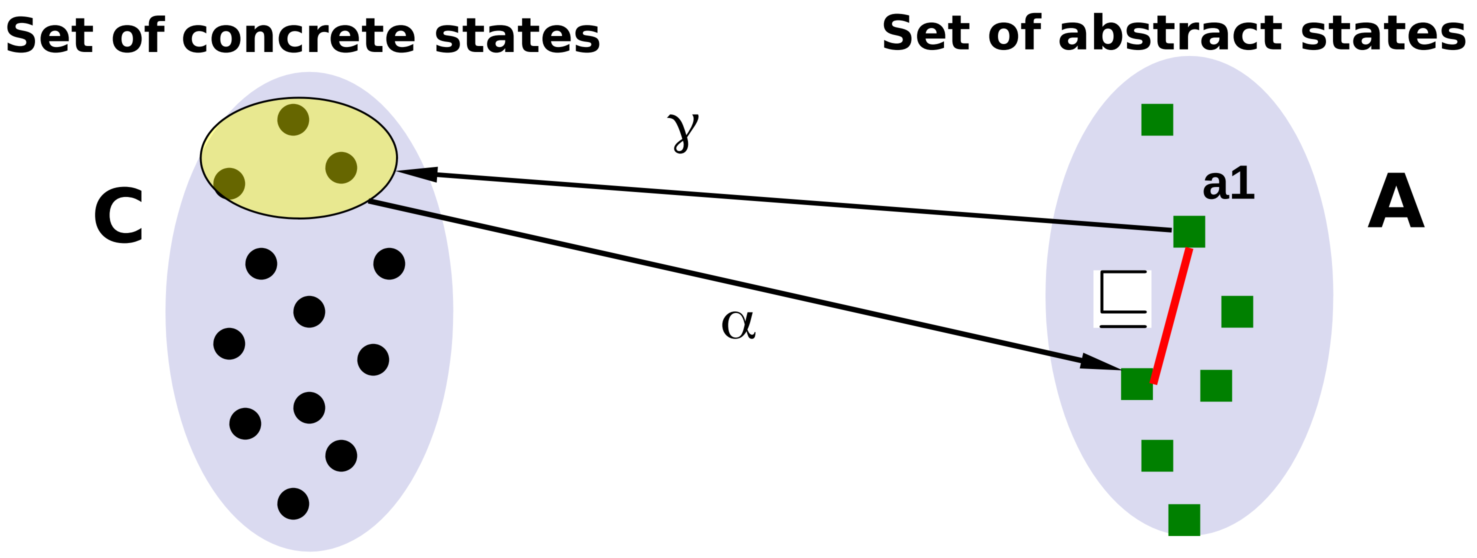
For all $S_1 \subseteq \mathcal{C}$ $\quad S_1 \subseteq \gamma(\alpha(S_1))$



Desirable Properties of α and γ

$$S_1 \subseteq \gamma(\alpha(S_1)) \quad \text{forall} \quad S_1 \subseteq \mathcal{C}$$

$$\alpha(\gamma(a_1)) \sqsubseteq a_1 \quad \text{forall} \quad a_1 \in \mathcal{A}$$

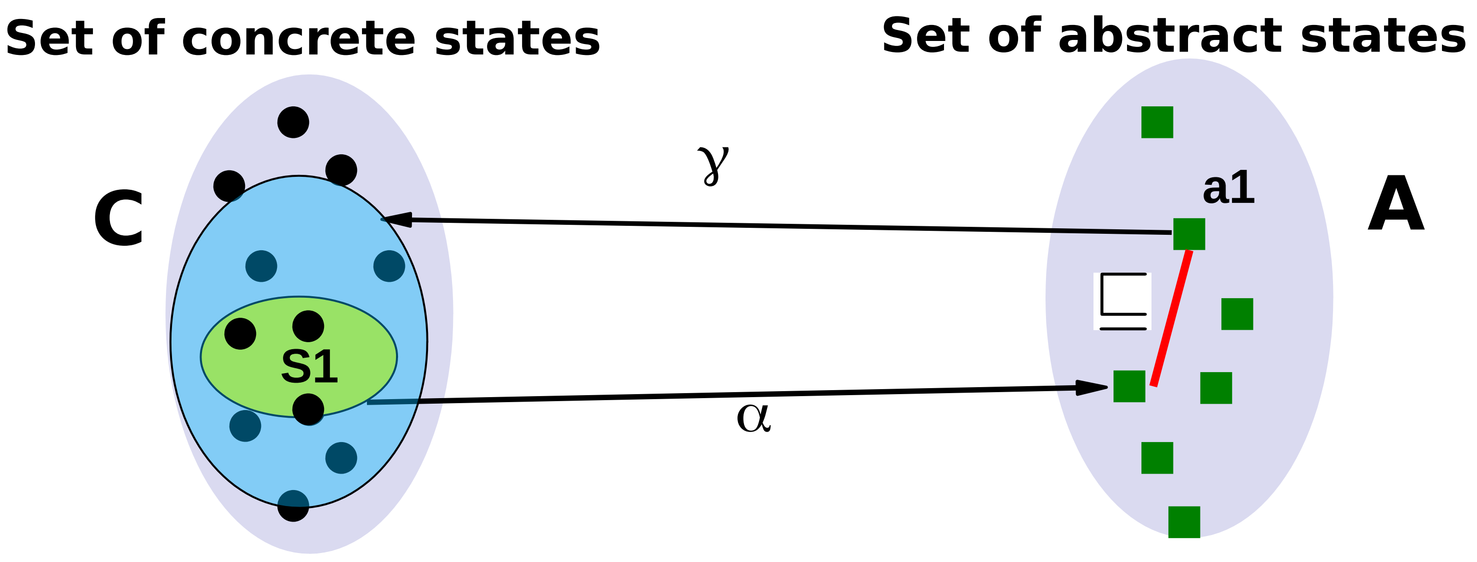


α and γ form a Galois connection

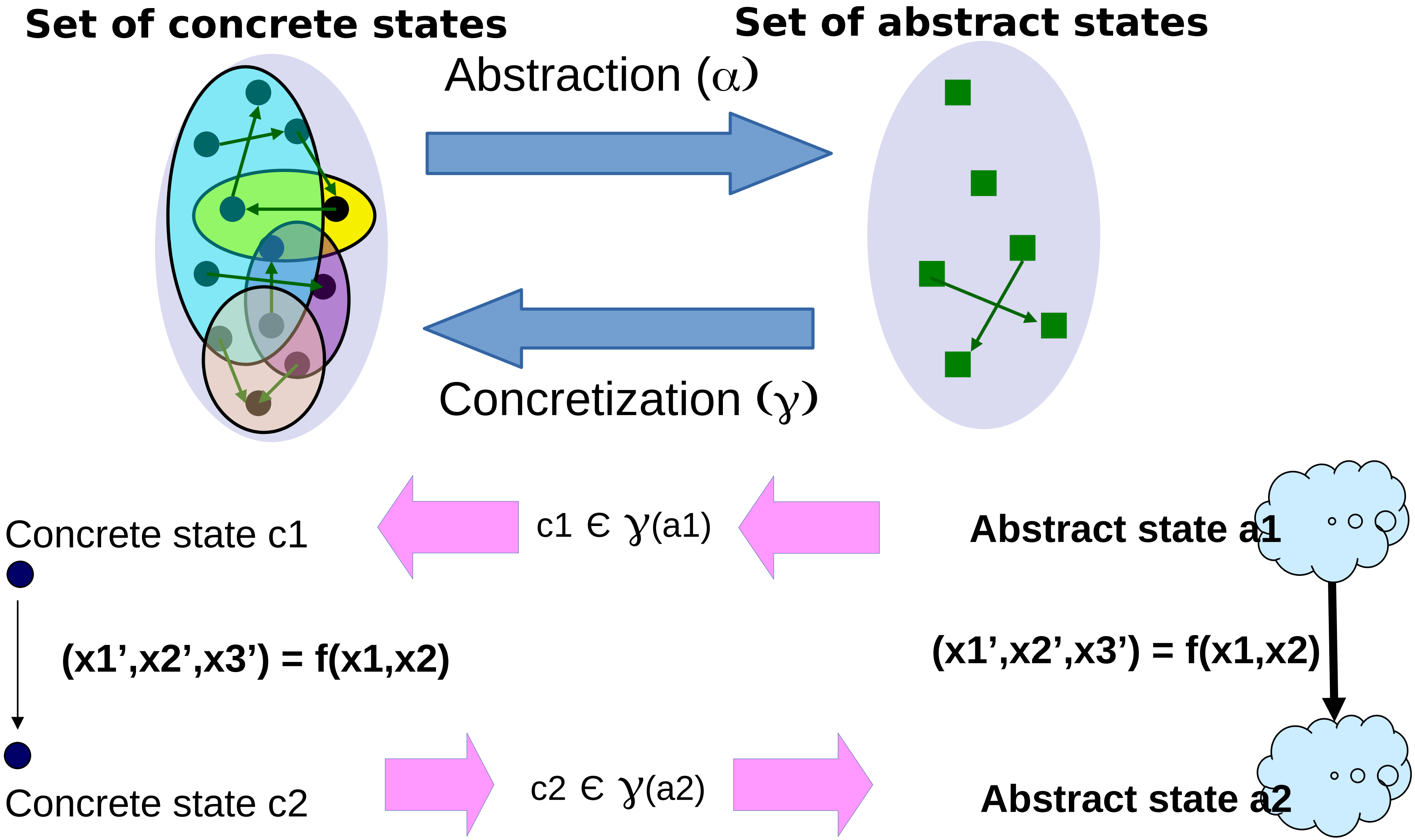
Desirable Properties of α and γ

- α and γ form a Galois connection
  - Second (equivalent) view:

$$\alpha(S_1) \sqsubseteq a_1 \;\Leftrightarrow\; S_1 \subseteq \gamma(a_1) \text{ for all } S_1 \subseteq S,\, a_1 \in \mathcal{A}$$



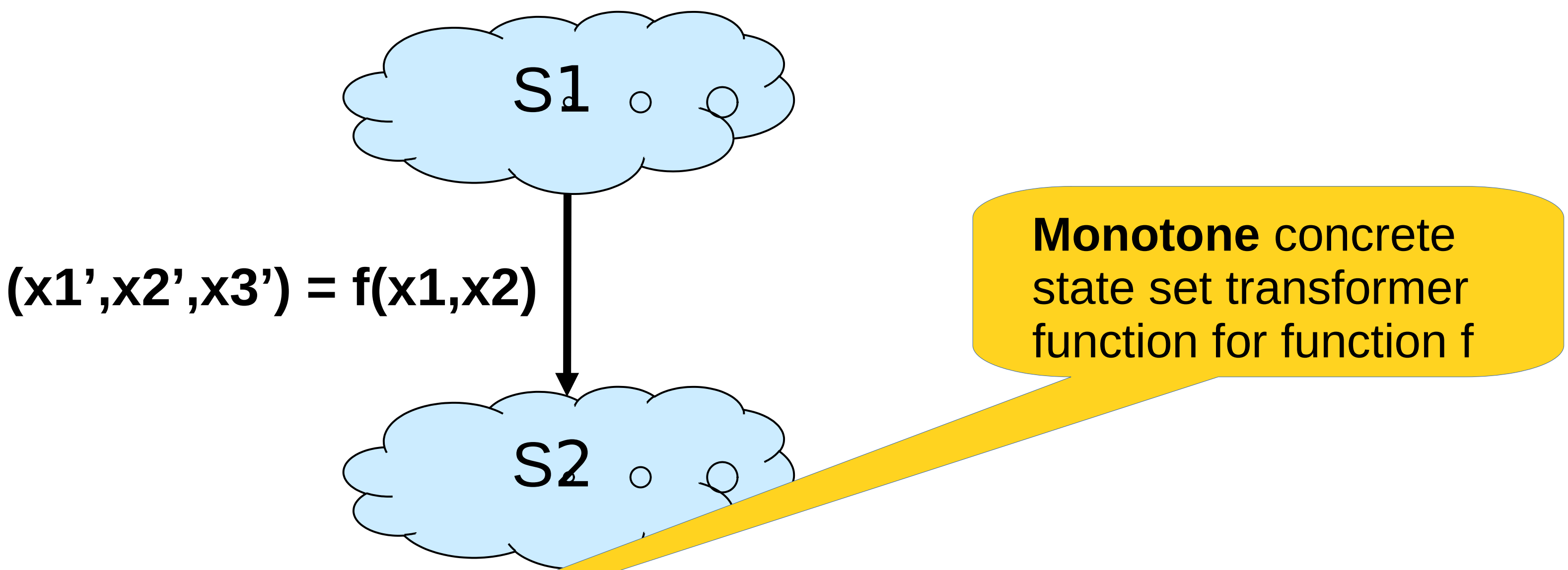
Computing Abstract State Transitions



Computing Abstract State Transitions

- Concrete state set transformer function
  - Example:

S1 = { (x1, x2, x3) | ….. }: set of concr. states

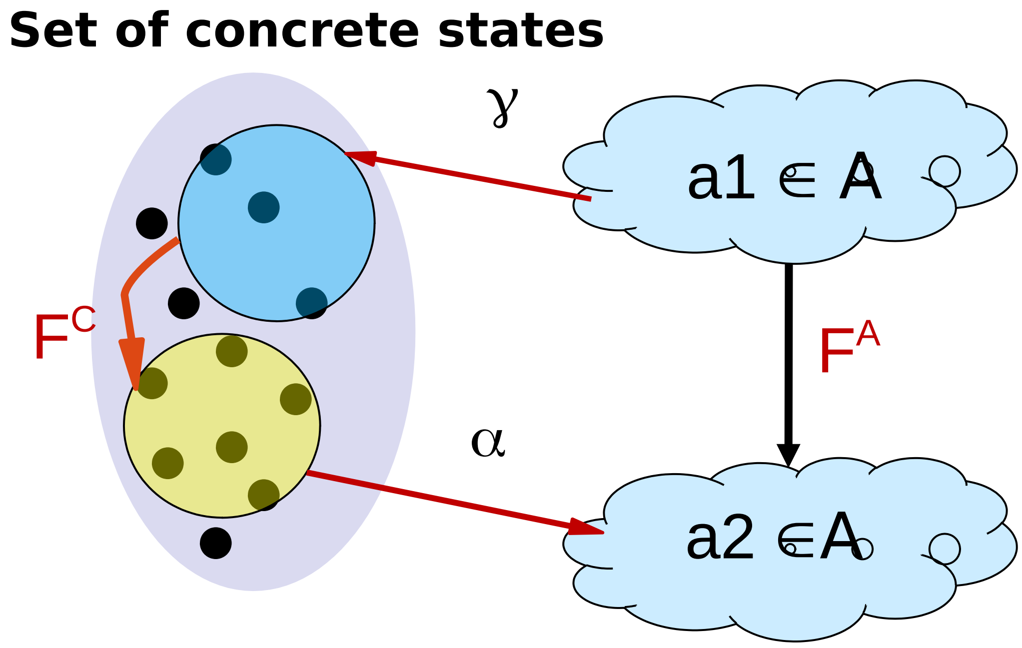


$$S2 = \{(x1', x2', x3') \mid \exists\ (x1, x2, x3) \in S1,\ (x1',x2',x3') = f(x1,x2)\}$$

$= F^C(S1)$: set of concrete states

Computing Abstract State Transitions

- Abstract state transformer function
  - Example:



$a2 = \alpha(F^C (\gamma\ (a1)))$ ideally, but $F^A(a1) \sqsupseteq \alpha(F^C (\gamma\ (a1)))$ often used

Summary

- Abstract interpretation is a general framework for analysis of state transition systems
- Widely used for verification and static analysis of programs
- Recent applications in neural network analysis
- Choice of right abstraction crucial to success
  - Balance between precision and efficiency

This lecture should help you understand the paper "An Abstract Domain for Certifying Neural Networks" by Singh et al. better