Knowledge Compilation for Boolean Functional Synthesis

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Joint work with S. Akshay, Jatin Arora, Ajith John, S. Krishna, Divya Raghunathan, Shetal Shah

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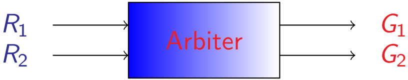

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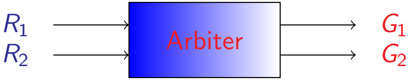

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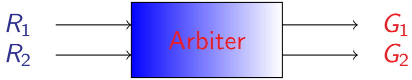

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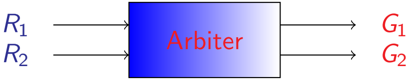

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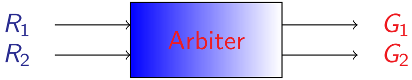

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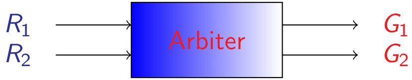

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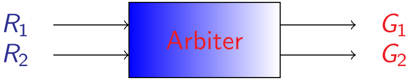

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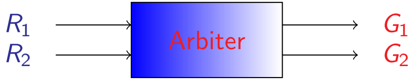

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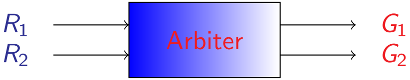

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Boolean Functional Synthesis

Synthesizing Boolean functions from a relational specification.

Boolean Functional Synthesis (BFnS)

Formal definition

Given Boolean relation $\varphi(x_1, .., x_n, y_1, .., y_m)$

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  - $F(X)$ must give right value of Y for all X s.t. $\exists Y \varphi(X, Y) = 1$
    - $F(X)$ inconsequential for other X

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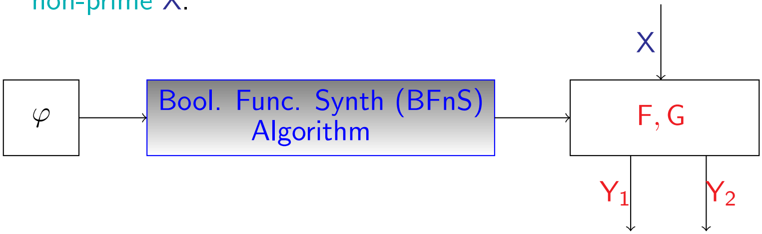

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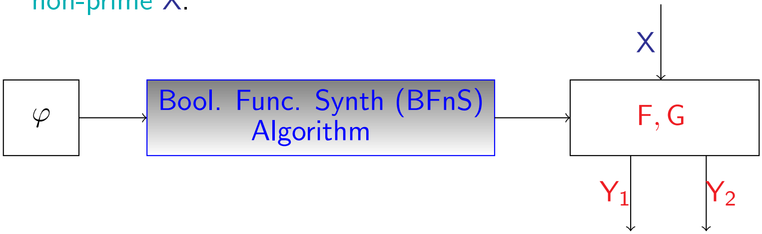

- For every non-prime X, finds non-trivial factors
- From prime X, values of $F(X)$ and $G(X)$ inconsequential.
  - $\exists Y_1, Y_2\ \varphi(X, Y_1, Y_2) = 0$ for such X.

Applications of Boolean Functional Synthesis

1. Cryptanalysis: Interesting but hard for synthesis!
2. Disjunctive decomposition of symbolic transition relations [Trivedi et al'02]
3. Quantifier elimination, of course!
   - $\exists Y\ \varphi(X, Y) \equiv \varphi(X, F(X))$
4. Certifying QBF-SAT solvers
   - Nice survey of applications by Shukla et al'19
5. Reactive controller synthesis
   - Synthesizing moves to stay within winning region
6. Program synthesis
   - Combinatorial sketching [Solar-Lezama et al'06, Srivastava et al'13]
   - Complete functional synthesis [Kuncak et al'10]
7. Repair/partial synthesis of circuits [Fujita et al'13]

Existing Approaches

1. Closely related to most general Boolean unifiers
   - Boole'1847, Lowenheim'1908, Macii'98
2. Extract Sk. functions from proof of validity of $\forall X \exists Y \varphi(X, Y)$
   - Bendetti'05, Jussilla et al'07, Balabanov et al'12, Heule et al'14
3. Using templates: Solar-Lezama et al'06, Srivastava et al'13
4. Self-substitution + function composition: Jiang'09, Trivedi'03
5. Synthesis from special normal form representation of specification
   - From ROBDDs: Tronci'98, Kukula et al'00, Kuncak et al'10, Fried et al'16, Tabajara et al'17
   - **From SynNNF:** Akshay et al'09
6. Incremental determinization: Rabe et al'17,'18
7. Quantifier instantiation techniques in SMT solvers
   - Barrett et al'15, Bierre et al'17
8. Input/output component separation: C. et al'18
9. Guess/learn Skolem function candidate + check + repair
   - John et al'15, Akshay et al'17,'18,'20, Golia et al'20

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- Boolean circuit (DAG)
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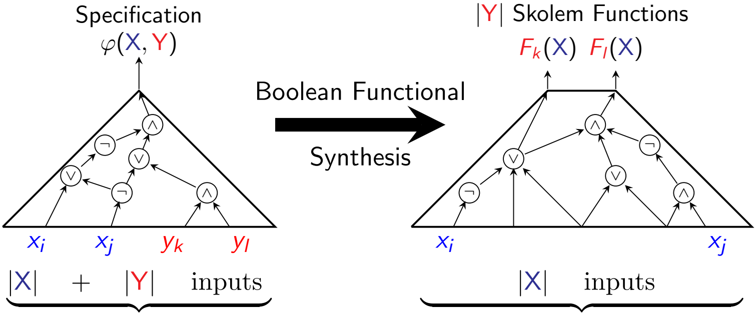
- Specification $\varphi(X, Y)$: $(|X| + |Y|)$-input, 1-output circuit
  - Other forms (ROBDD/CNF/DNF ...) efficiently converted
- Desired Sk. fn. vector $F(X)$: $|X|$-input, $|Y|$-output circuit
  - No additional restrictions (ROBDD/CNF/DNF ...)



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- Bad news: [CAV2018]
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Efficient algorithms for Boolean functional synthesis unlikely

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  - Efficient synthesis doesn't require full capabilities of these stronger normal forms.

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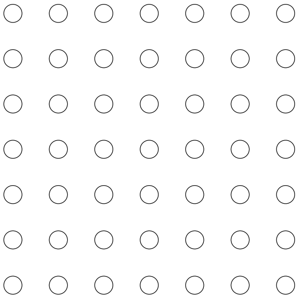

— Set of all valuations of X.

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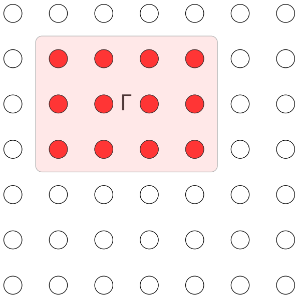

— Can't set y to 1 to satisfy φ: $\Gamma(X) \triangleq \neg\varphi(X, y)[y \mapsto 1]$

E.g. If $\varphi \equiv (x_1 \vee y) \wedge (x_1 \vee x_2 \vee \neg y)$, then
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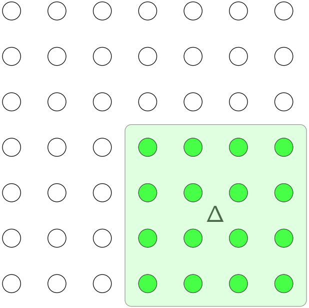

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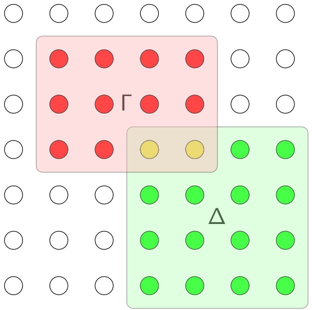

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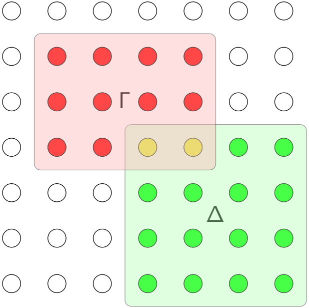

Lemma [Trivedi'03, Jiang'09,Fried et al'16]

Every Skolem function for y in φ must

- Evaluate to 1 in $(\Delta \setminus \Gamma)$ and to 0 in $(\Gamma \setminus \Delta)$
- Be an **interpolant** of $(\Delta \setminus \Gamma)$ and $(\Gamma \setminus \Delta)$

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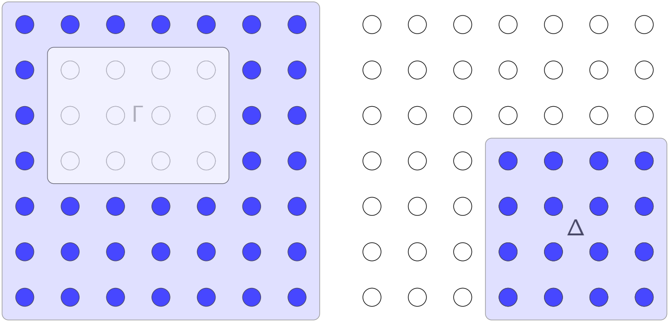

— Specific interpolants of $(\Delta \setminus \Gamma)$ & $(\Gamma \setminus \Delta)$

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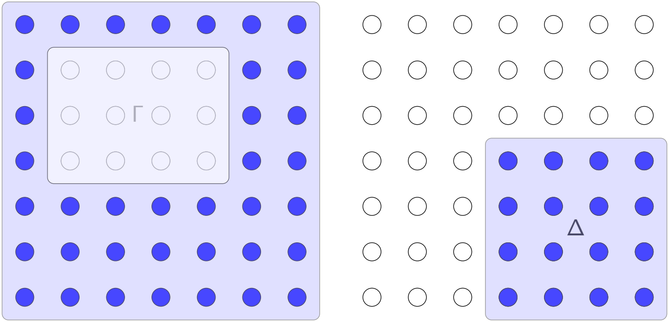

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How do we compute

$\exists y_1, \ldots y_i\, \varphi(X, y_1, \ldots y_i, F_{i+1}(X), \ldots F_m(X))$ **efficiently?**

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- Tronci'98, Kukula et al'00, Kuncak et al'10, Fried et al'16, Tabajara et al'17
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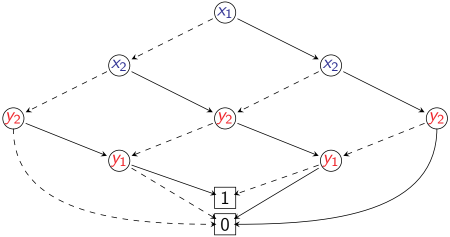

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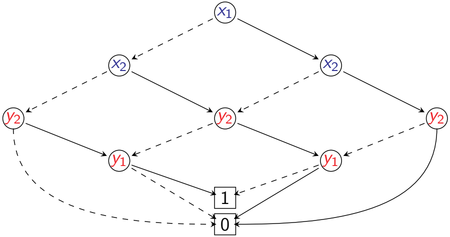
$\exists y_1 \varphi(X, Y) = y_2$

$F_2(X) = 1$



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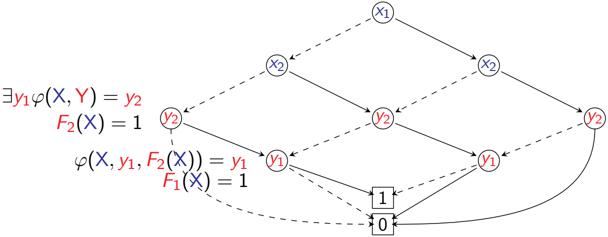

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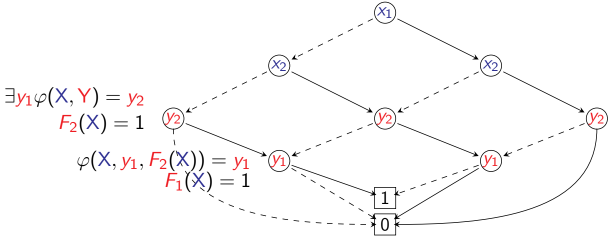

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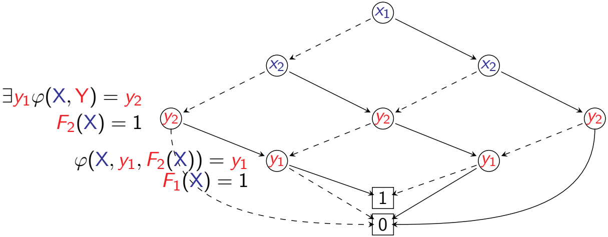

- Significant optimizations in Fried et al'16, Tabajara et al'17
- Spec ROBDD can be exp. larger with input-first ordering
  - $\varphi(X, Y) \equiv \bigwedge_{i=1}^{n} (x_i \Leftrightarrow y_i)$
  - Size $\Omega(2^n)$ with input-first ordering, $\Theta(n)$ with interleaved input-output ordering,

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- What if the best variable order for the specification is not input-first?
    - Can we synthesize efficiently for such specs?

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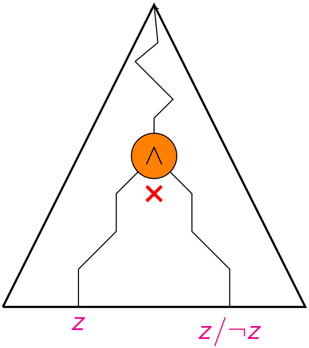
$\varphi(X, Y)$ in DNNF except on X



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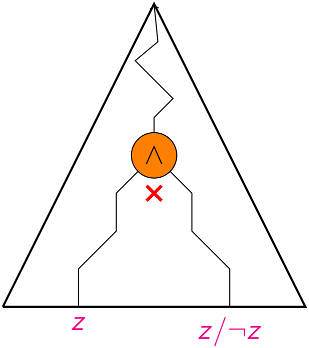

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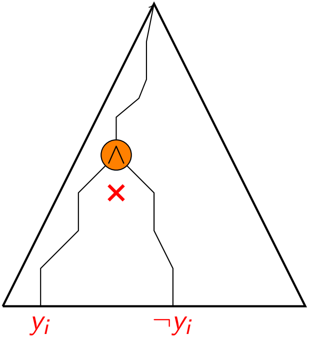

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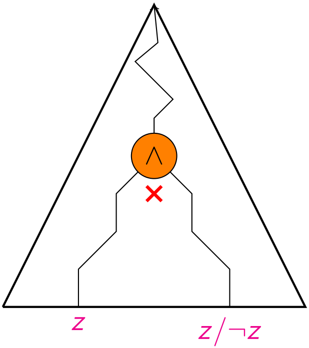

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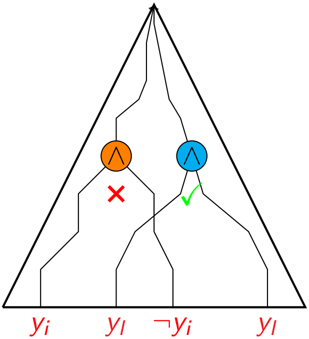

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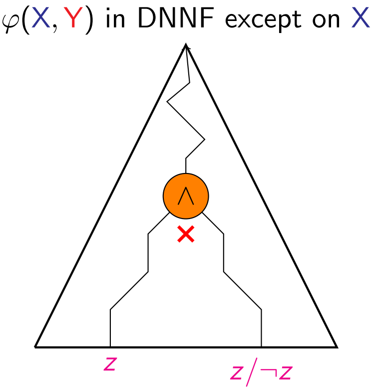

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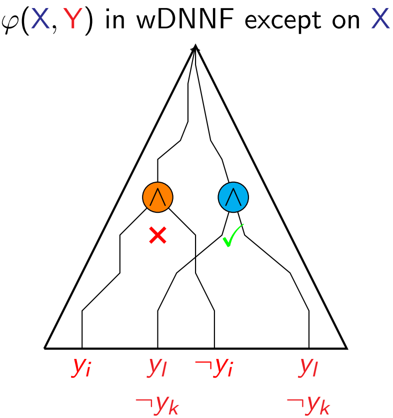

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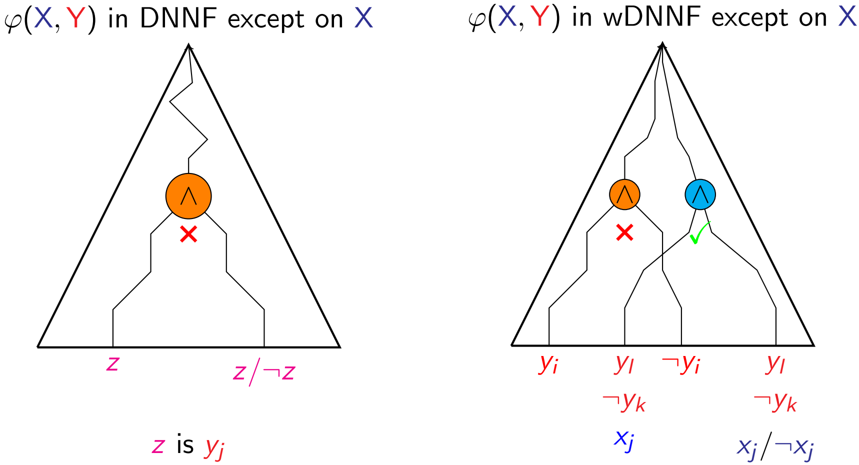

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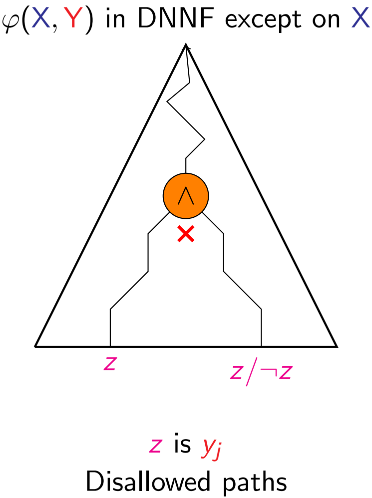

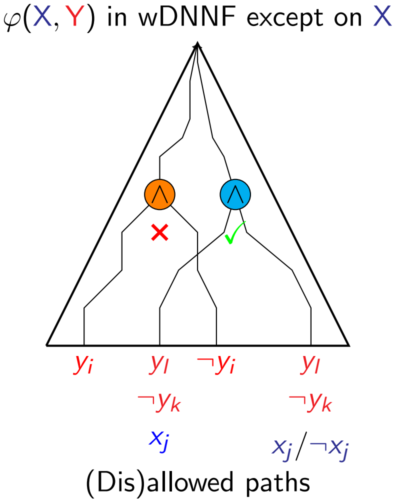

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- **Practical performance promising!**

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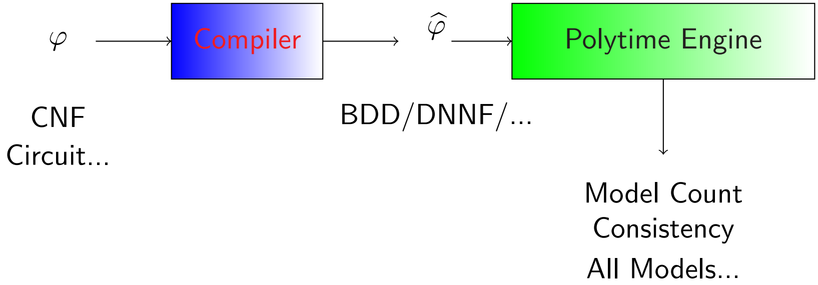

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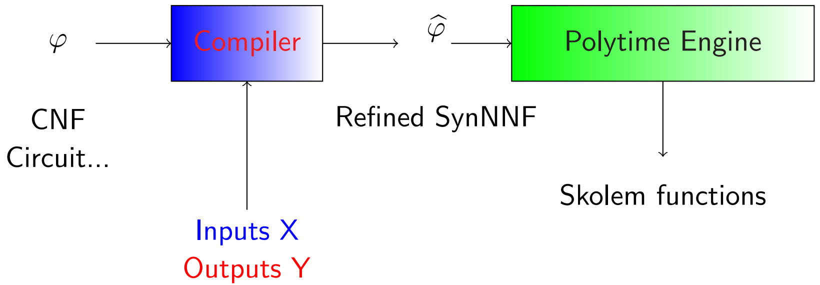
Our Definition

… a family of approaches for addressing the intractability of synthesis problems. A propositional model is compiled in an off-line phase in order to support some queries in polytime.



Towards a normal form for efficient synthesis

- Represent $\varphi(x_1, .., x_n, y_1, .., y_m)$ as NNF DAG
  - Boolean circuit, $\wedge$ and $\vee$ at internal nodes, $\neg$ only at leaves

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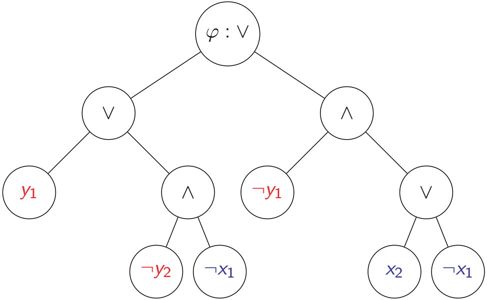

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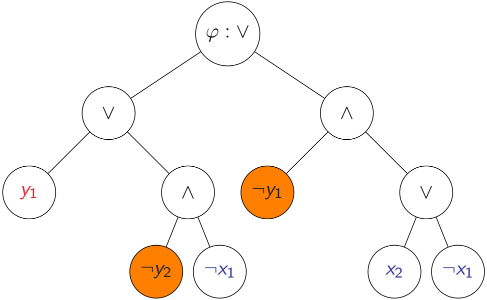

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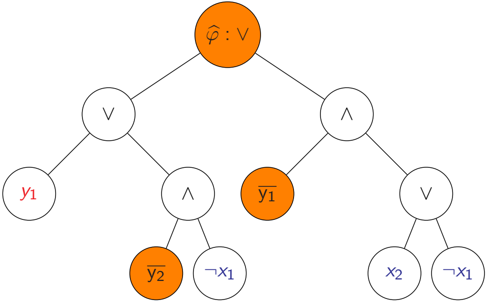
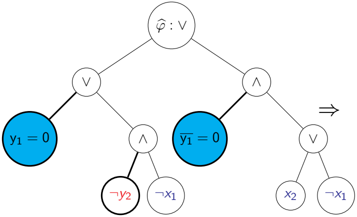
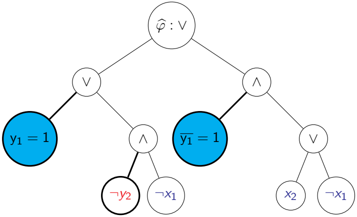


- **Positive form of specification:**
  $\widehat{\varphi}(x_1, \ldots x_n,\ y_1, \ldots y_m,\ \overline{y_1}, \ldots \overline{y_m})$
  - Monotone w.r.t all y_i and $\overline{y_i}$

Simple properties of $\widehat{\varphi}$



$\Rightarrow \quad \exists y_1 \varphi \quad \Rightarrow$



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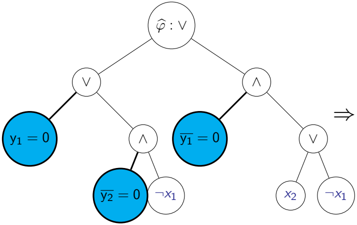
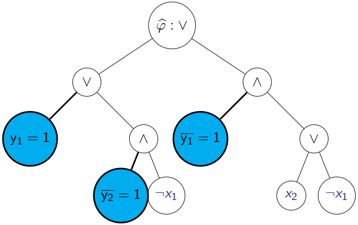


$\Rightarrow \exists y_1 y_2 \varphi \Rightarrow$



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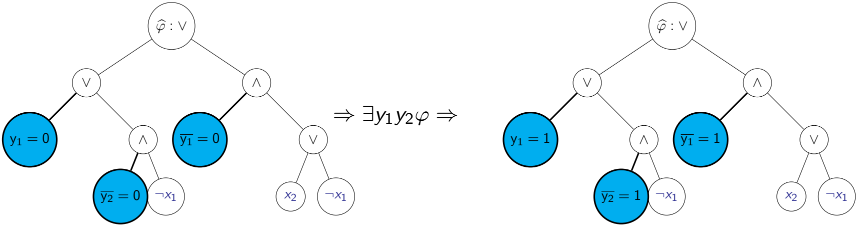

- $\widehat{\varphi}(x_1 \ldots x_n, \overbrace{0..0}^{i}, y_{i+1} \ldots y_m, \overbrace{0..0}^{i}, \neg y_{i+1} \ldots \neg y_m) \Rightarrow \exists y_1 \ldots y_i \; \varphi(\ldots)$
- $\widehat{\varphi}(x_1 \ldots x_n, \overbrace{1..1}^{i}, y_{i+1} \ldots y_m, \overbrace{1..1}^{i}, \neg y_{i+1} \ldots \neg y_m) \Leftarrow \exists y_1 \ldots y_i \; \varphi(\ldots)$

Iterated reducts of $\widehat{\varphi}$

Given

- Positive form of spec $\widehat{\varphi}(x_1, \ldots x_n, y_1, \ldots y_m, \overline{y_1}, \ldots \overline{y_m})$
- Linear order of outputs $y_1 \prec \cdots \prec y_m$

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Define $[\widehat{\varphi}]_i$ as

$$\widehat{\varphi}(x_1, \ldots x_n, \overbrace{1..1}^{i-1}, \ \mathrm{y_i}, \ y_{i+1} \ldots y_m, \overbrace{1..1}^{i-1}, \ \overline{\mathrm{y_i}}, \ \neg y_{i+1} \ldots \neg y_m).$$

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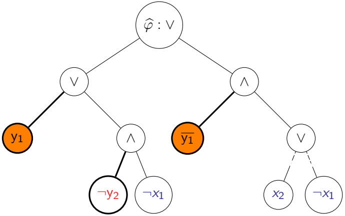

$[\widehat{\varphi}]_1(x_1, x_2, y_1, y_2, \overline{y_1})$

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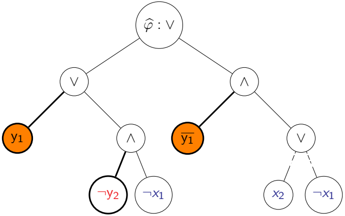

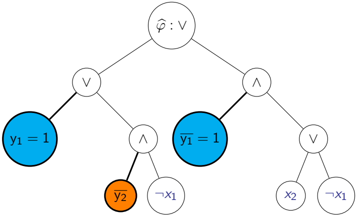

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$[\widehat{\varphi}]_2(x_1, x_2, y_2, \overline{y_2})$

Iterated reducts and existential quantification

Already seen $\exists y_1 \varphi(X, Y) \;\Rightarrow\; \widehat{\varphi}\,|_{y_1=1,\overline{y_1}=1}$

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- $\exists y_1 \varphi(X, Y) \Leftrightarrow [\widehat{\varphi}]_1\,|_{y_1=1,\overline{y_1}=1}$ for all $X, y_2, \ldots y_m$ iff $\neg\exists X, y_2, \ldots y_m \left([\widehat{\varphi}]_1 \Leftrightarrow y_1 \wedge \overline{y_1}\right)$.

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- Inductively, $\exists y_1, \ldots y_i \varphi(X, Y) \;\Leftrightarrow\; [\widehat{\varphi}]_i\,|_{y_i=1,\overline{y_i}=1}$ iff $\neg\exists X, y_{i+1}, \ldots y_m \left([\widehat{\varphi}]_i \;\Leftrightarrow\; y_i \wedge \overline{y_i}\right)$.

SynNNF: A negation normal form for efficient synthesis

An NNF specification $\varphi(X, Y)$ is in

SynNNF w.r.t. a linear order $y_1 \prec y_2 \prec \cdots \prec y_m$ iff

$\neg\exists x_1, \ldots x_n, y_{i+1} \ldots y_m \quad (\; [\widehat{\varphi}]_i \Leftrightarrow \; \mathrm{y_i} \wedge \overline{\mathrm{y_i}} \;)$ for all $i \in \{1, \ldots n\}$

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Can $[\widehat{\varphi}]_i$ be made to "behave like" $\mathrm{y_i} \wedge \overline{\mathrm{y_i}}$ for any i?

If yes, φ is not in SynNNF; else it is in SynNNF

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Skolem fn for y_i (in terms of $y_{i+1}, \ldots y_m, \mathrm{X}$)

- $\exists y_1, \ldots y_{i-1}\ \varphi(\mathrm{X}, y_1, \ldots y_{i-1}, 1, y_{i+1}, \ldots y_m)$

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Observations:

- Not purely structural restriction on representation of φ

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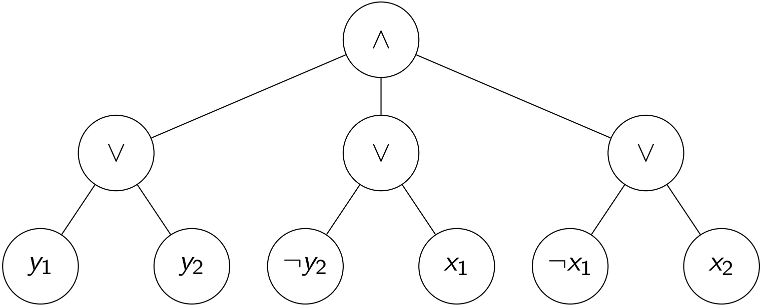
Observations:

- Not purely structural restriction on representation of φ
- Reminiscent of Deterministic DNNF (dDNNF)
    - For every $\vee$ node representing $\varphi_1 \vee \varphi_2$, require $\varphi_1 \wedge \varphi_2 = \bot$.

SynNNF: An Example

$$\varphi(x_1, x_2, y_1, y_2) = (y_1 \vee y_2) \wedge (\neg y_2 \vee x_1) \wedge (\neg x_1 \vee x_2)$$



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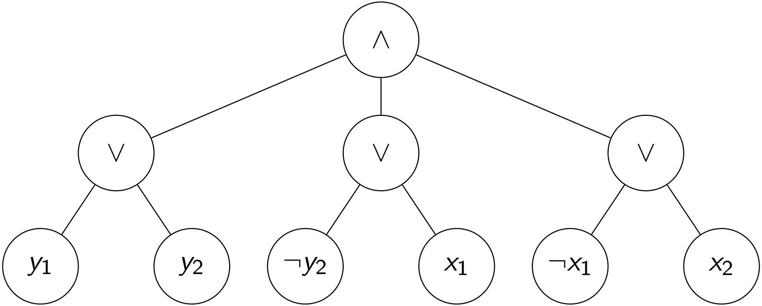

Representation of φ **not** in DNNF/wDNNF

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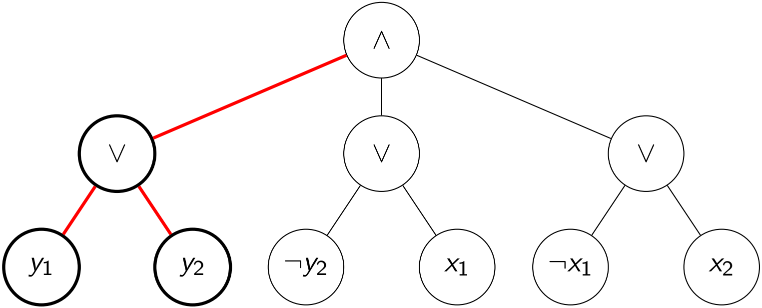
Output sequence: $\underline{y_1} \prec y_2$



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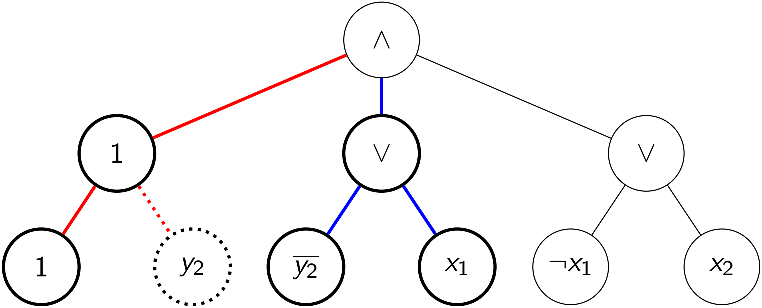

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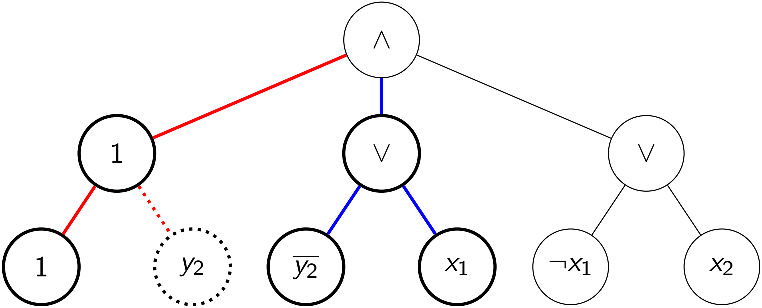
$\varphi(x_1, x_2, y_1, y_2) = (y_1 \vee y_2) \wedge (\neg y_2 \vee x_1) \wedge (\neg x_1 \vee x_2)$

Output sequence: $y_1 \prec \underline{y_2}$



Representation of φ in SynNNF **w.r.t** $y_1 \prec y_2$

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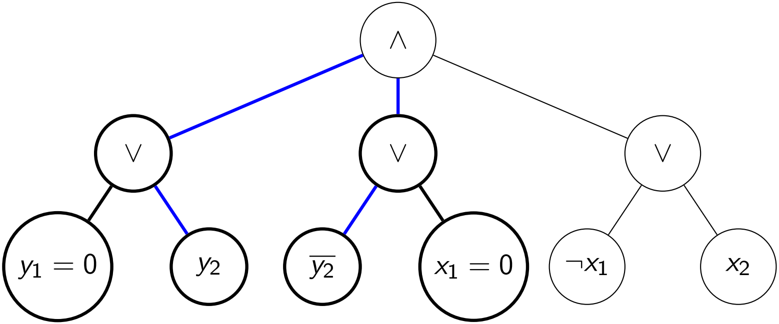

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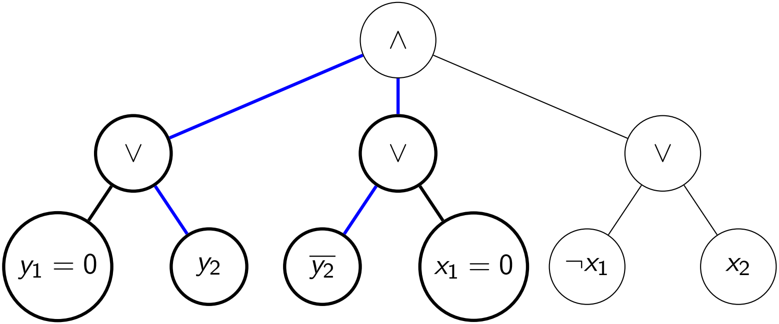
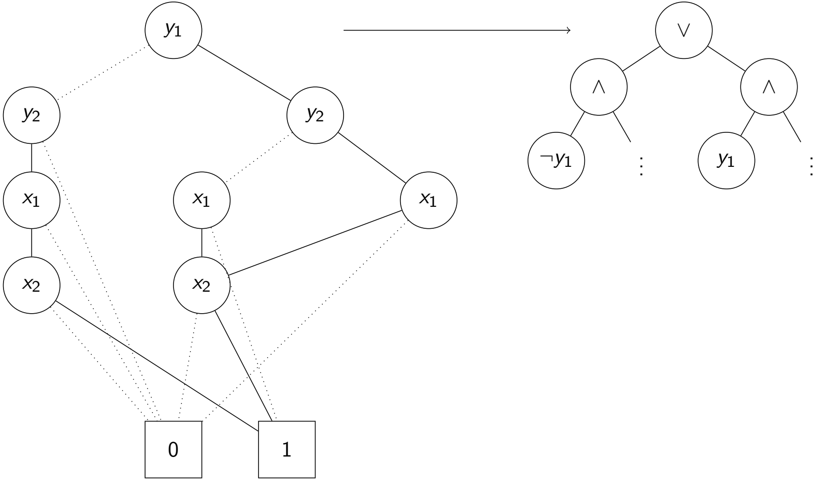
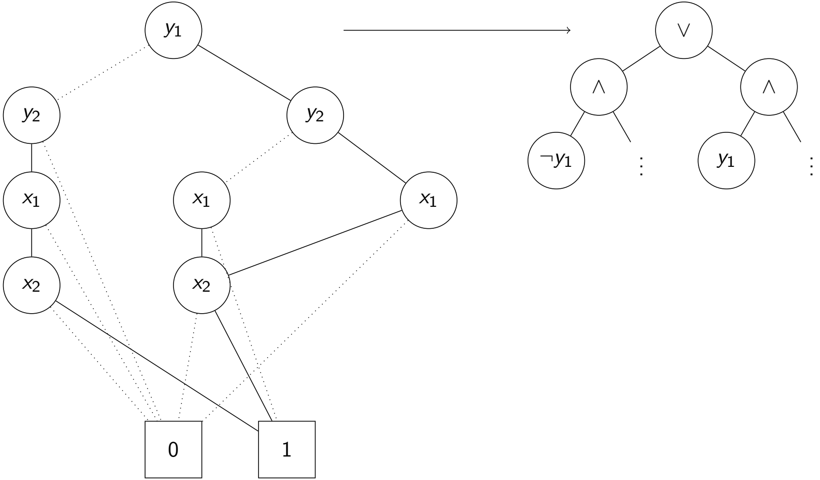


Representation of φ not in SynNNF **w.r.t** $y_2 \prec y_1$

BDD and SynNNF

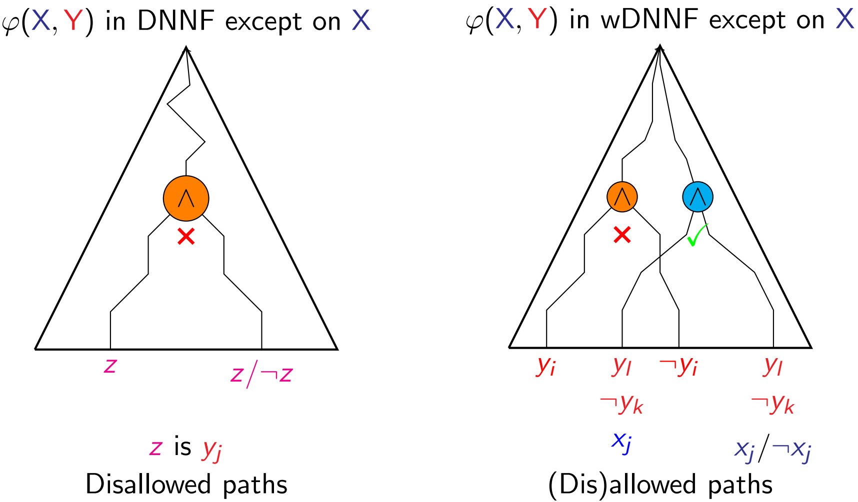


BDD and SynNNF

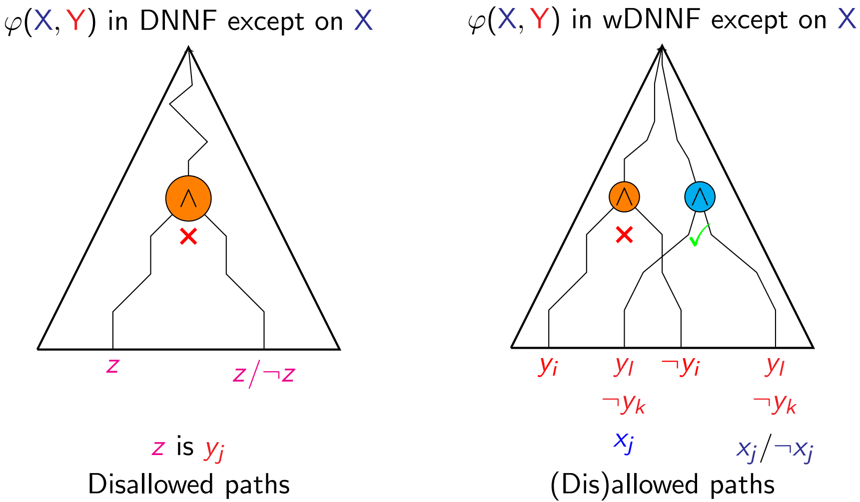


BDD → SynNNF in linear time for any output order ≺ and any BDD variable order.

DNFF, wDNNF and SynNNF



DNNF, wDNNF and SynNNF



A specification in DNNF or wDNNF is already in SynNNF for any output order $\prec$.

SynNNF: Some observations

- Every propositional formula representable in SynNNF for every ordering of outputs
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- Given φ, checking if φ is in SynNNF w.r.t. any (unspecificed) order $\prec$ is in Σ_2^P.

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NNF $\sqsubset$ SynNNF $\sqsubset$ DNNF $\sqsubset$ dDNNF $\sqsubset$ BDD

Operations with SynNNF

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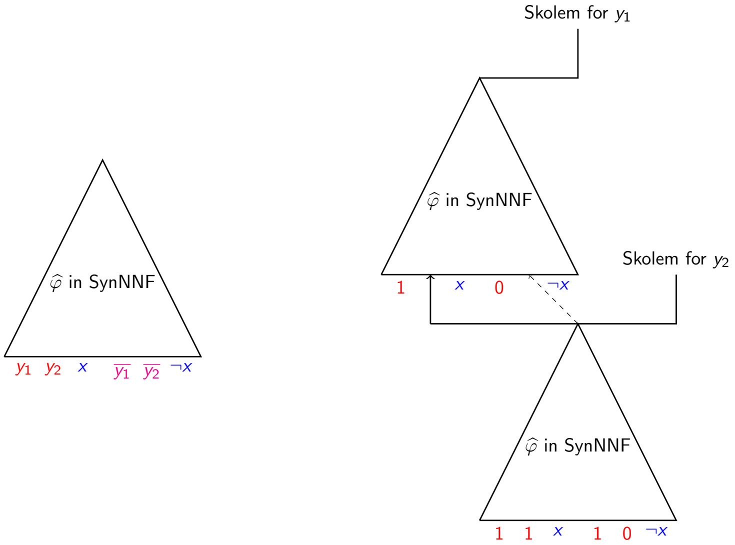
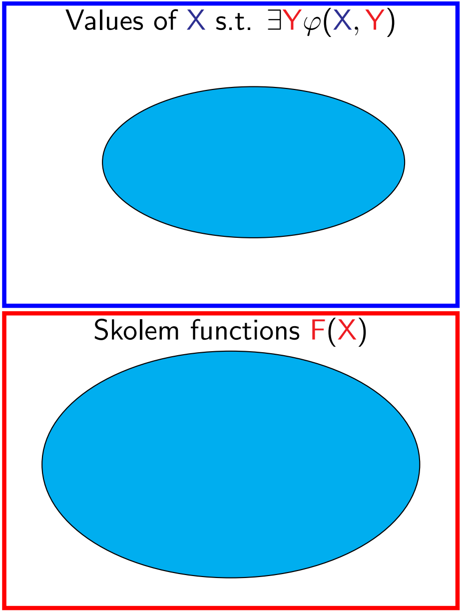
- Computing $\varphi_1 \wedge \varphi_2$ in SynNNF in poly-time not possible unless P = NP
- Computing $\varphi_1 \vee \varphi_2$ in SynNNF in same ordering of Y takes constant time
- Existentially quantifying $y_1, \ldots y_m$ takes linear time.
    - Quantifying subset of Y not possible in linear time in general.

How does SynNNF help Skolem function synthesis?



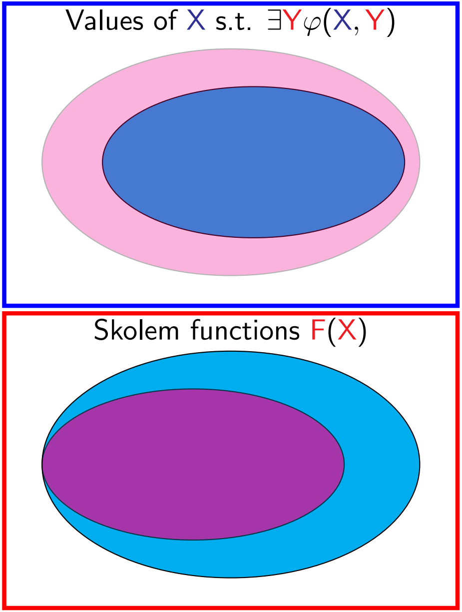
Synthesis: $m \times |\varphi|$ circuit size, $\mathcal{O}(m^2)$ additional wires.

Refinement w.r.t. synthesis



Given spec: $\varphi(X, Y)$

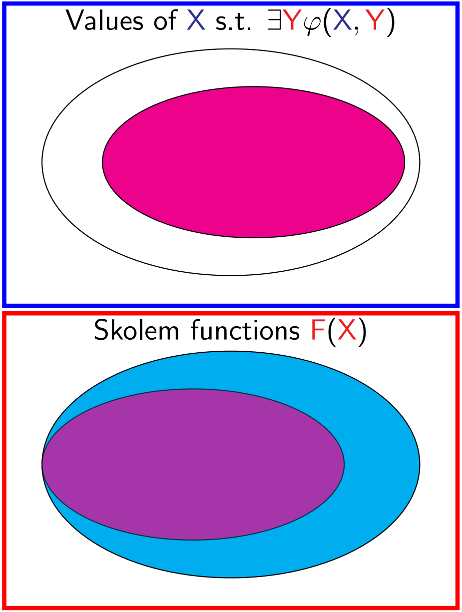
Refinement w.r.t. synthesis



Given spec: $\varphi(X, Y)$

Refined spec: $\widetilde{\varphi}(X, Y)$

$\widetilde{\varphi} \preceq_{syn} \varphi$



Given spec: $\varphi(X, Y)$

Strongly
Refined spec: $\widetilde{\varphi}(X, Y)$
$\widetilde{\varphi} \preceq^*_{syn} \varphi$

Refinement w.r.t. synthesis

Lemma

If $\widetilde{\varphi}(X, Y) \preceq_{syn} \varphi(X, Y)$, every Skolem function vector for Y in $\widetilde{\varphi}$ is also a Skolem function vector for Y in φ.

Refinement w.r.t. synthesis

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Example: $(y_2 \wedge y_1) \preceq_{syn}$
$((\neg y_1 \vee y_2 \vee x_1) \wedge (y_1 \vee \neg y_2) \wedge (y_1 \vee \neg x_1) \wedge (y_2 \vee x_2))$

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3. If $\bigwedge_{y_i \in \mathsf{Y}} (F|_{y_i=0} \Leftrightarrow F|_{y_i=1})$, then $1 \preceq_{syn} F$ and $F|_{\mathsf{Y}=\mathsf{a}} \preceq^*_{syn} F$, where a is any vector in $\{0,1\}^m$.

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    1. Let $\widetilde{F}_1 \preceq^*_{syn} F_1$ and $\widetilde{F}_2 \preceq^*_{syn} F_2$. Then $(\widetilde{F}_1 \vee \widetilde{F}_2) \preceq^*_{syn} (F_1 \vee F_2)$.

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   2. Let $\widetilde{F_1} \preceq_{syn} F_1$ and $\widetilde{F_2} \preceq_{syn} F_2$. If the output supports of F_1 and F_2, and similarly of $\widetilde{F_1}$ and $\widetilde{F_2}$, are disjoint, then $(\widetilde{F_1} \wedge \widetilde{F_2}) \preceq_{syn} (F_1 \wedge F_2)$. If, in addition, $\widetilde{F_1} \preceq^*_{syn} F_1$ and $\widetilde{F_2} \preceq^*_{syn} F_2$, then $(\widetilde{F_1} \wedge \widetilde{F_2}) \preceq^*_{syn} (F_1 \wedge F_2)$.

Putting it all together

Tool C2Syn:

- Input: φ in CNF (or AIG)
- Output: Refined $\widetilde{\varphi}$ in SynNNF

- Branches only on output variables
- Aggressively tries to refine whenever possible
- Details in our FMCAD 2019 paper

Experimental Results

Comparison of run-time with

- CADET [Rabe et al 2016]
- BFSS [Akshay et al 2018]
- BDD [BDD pipeline of BFSS]

Benchmarks: QBFEVAL 2018 and Factorization (408 total)

Benchmarks (Total)	Compiled By C2Syn			BDD compilation	Total in SynNNF
	Stage I	Stage II	Total		
QBFEVAL (402)	103	83	186	153	**283**
FA.QD (6)	0	6	6	6	**6**

Table: Compilation into SynNNF

Bench mark	C2Syn vs CADET		C2Syn vs BFSS		C2Syn \ (CADET ∪ BFSS)
	C2Syn\ CADET	CADET\ C2Syn	C2Syn\ BFSS	BFSS\ C2Syn	
QBFEVAL	78	105	83	78	**75**
FA.QD	2	0	3	0	**2**

Table: Comparison Results of C2Syn

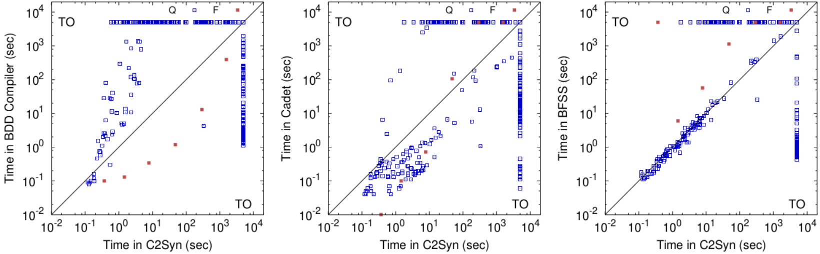

Fig. 1: Time comparisons: C2Syn vs BDD^{BFSS}, CADET, BFSS

Conclusion

- SynNNF: A new normal form for polynomial-time synthesis
- Refinement w.r.t. synthesis useful in practice
  - Formalization of folklore
  - CNF $\rightarrow$ Refined SynNNF much more efficient than CNF $\rightarrow$ SynNNF
- Experimental results with preliminary implementation show promise
- It appears that SynNNF can be further weakened to achieve poly-time synthesis
  - Ongoing work