Algorithms for Boolean Functional Synthesis

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Joint work with S. Akshay, Jatin Arora, Ajith John, S. Krishna, Divya Raghunathan, Shetal Shah

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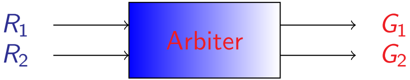

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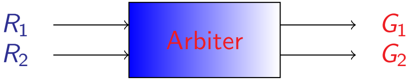

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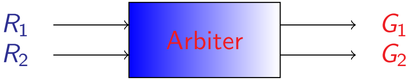

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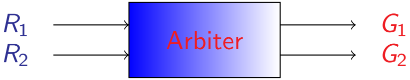

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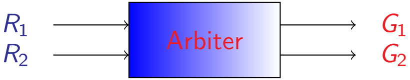

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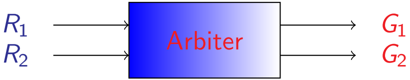

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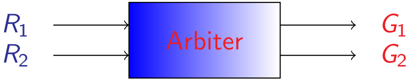

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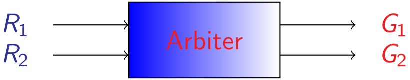

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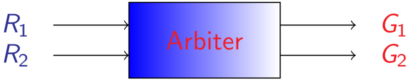

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Boolean Functional Synthesis

Synthesizing Boolean functions from a relational specification.

Boolean Functional Synthesis (BFnS)

Formal definition

Given Boolean relation $\varphi(x_1, .., x_n, y_1, .., y_m)$

- x_i *input* variables (vector X)
- y_j *output* variables (vector Y)

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  - $F(X)$ must give right value of Y for all X s.t. $\exists Y \varphi(X, Y) = 1$
    - $F(X)$ inconsequential for other X

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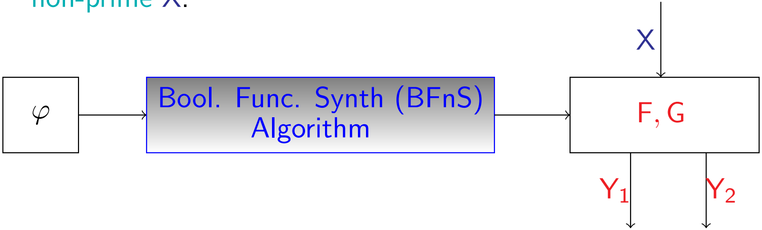

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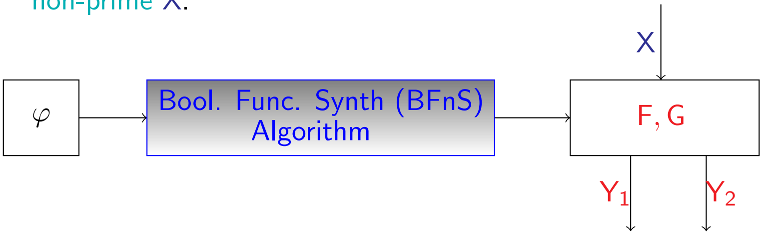

- For every non-prime X, finds non-trivial factors
- From prime X, values of $F(X)$ and $G(X)$ inconsequential.
  - $\exists Y_1, Y_2\ \varphi(X, Y_1, Y_2) = 0$ for such X.

Applications of Boolean Functional Synthesis

1. Cryptanalysis: Interesting but hard for synthesis!
2. Disjunctive decomposition of symbolic transition relations [Trivedi et al'02]
3. Quantifier elimination, of course!
   - $\exists Y\ \varphi(X, Y) \equiv \varphi(X, F(X))$
4. Certifying QBF-SAT solvers
   - Nice survey of applications by Shukla et al'19
5. Reactive controller synthesis
   - Synthesizing moves to stay within winning region
6. Program synthesis
   - Combinatorial sketching [Solar-Lezama et al'06, Srivastava et al'13]
   - Complete functional synthesis [Kuncak et al'10]
7. Repair/partial synthesis of circuits [Fujita et al'13]

Existing Approaches

1. Closely related to most general Boolean unifiers
    - Boole'1847, Lowenheim'1908, Macii'98
2. Extract Sk. functions from proof of validity of $\forall X \exists Y \varphi(X, Y)$
    - Bendetti'05, Jussilla et al'07, Balabanov et al'12, Heule et al'14
3. Using templates: Solar-Lezama et al'06, Srivastava et al'13
4. Self-substitution + function composition: Jiang'09, Trivedi'03
5. Synthesis from special normal form representation of specification
    - From ROBDDs: Kukula et al'00, Kuncak et al'10, Fried et al'16, Tabajara et al'17
    - From SynNNF: Akshay et al'09
6. Incremental determinization: Rabe et al'17,'18
7. Quantifier instantiation techniques in SMT solvers
    - Barrett et al'15, Bierre et al'17
8. Input/output component separation: C. et al'18
9. Guess/learn Skolem function candidate + check + repair
    - John et al'15, Akshay et al'17,'18,'20, Golia et al'20

How Hard (or Easy) Is BFnS?

- Boolean circuit: DAG with AND-, OR-, NOT-labeled nodes
- Input: $\varphi(X, Y)$ as $(|X| + |Y|)$-input, 1-output circuit
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  - Skolem functions need not be monotone
  - Different argument for lower bounds on Skolem circuits

Some Good and Bad News

Bad news: [CAV2018]

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- If φ is represented in special normal form, synthesis solvable in polynomial (in $|\varphi|$) time and space.
  - Synthesis Negation Normal Form (SynNNF)
    - Talk in "Beyond Satisfiability" workshop on Mar 23
  - Reasonably common in practice

Experiments: Guess-check-repair algorithms work well in practice

Overview of Guess-Check-Repair Paradigm

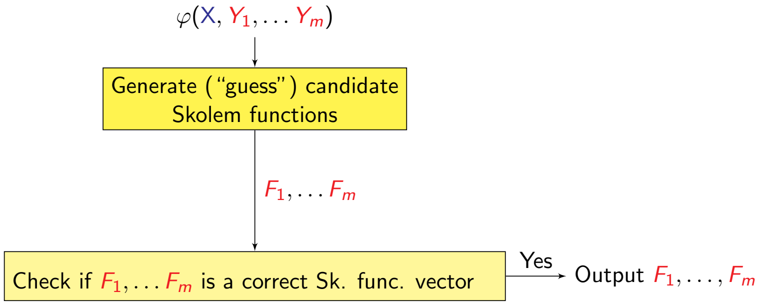
$$\varphi(X, Y_1, \ldots Y_m)$$

$\downarrow$

Generate ("guess") candidate
Skolem functions

$\downarrow$ $F_1, \ldots F_m$

Overview of Guess-Check-Repair Paradigm



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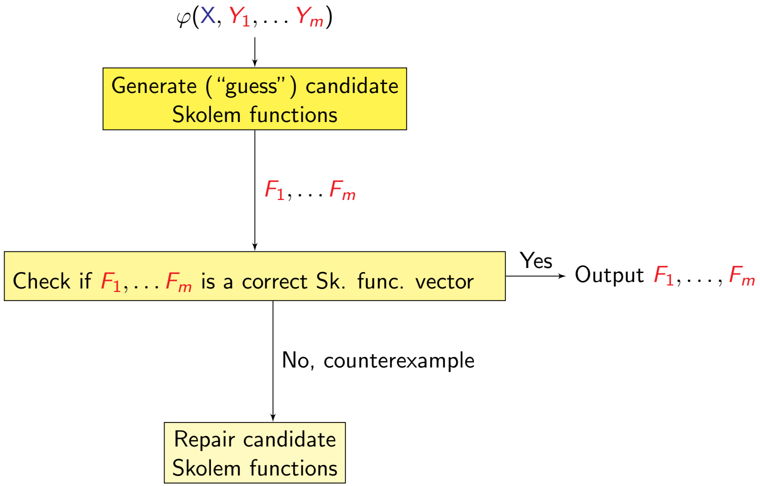

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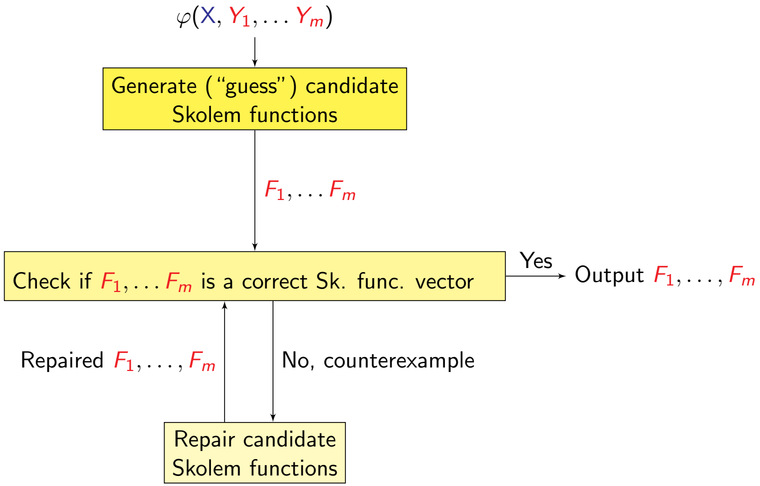

"Guess"-ing candidate Skolem functions ($|Y| = 1$)

Find $F(X)$ such that $\exists y\ \varphi(X, y) \equiv \varphi(X, F(X))$

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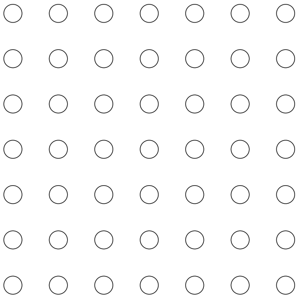

— Set of all valuations of X.

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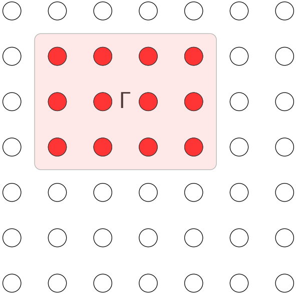

— Can't set y to 1 to satisfy φ: $\Gamma(X) \ \triangleq\ \neg\varphi(X, y)[y \mapsto 1]$

E.g. If $\varphi \equiv (x_1 \vee y) \wedge (x_1 \vee x_2 \vee \neg y)$, then
$\Gamma(X) \ = \ \neg((x_1 \vee 1) \wedge (x_1 \vee x_2 \vee 0)) \ = \ \neg(x_1 \vee x_2) \ = \ \neg x_1 \wedge \neg x_2$

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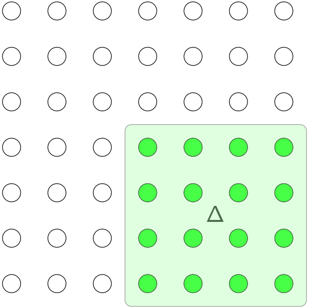

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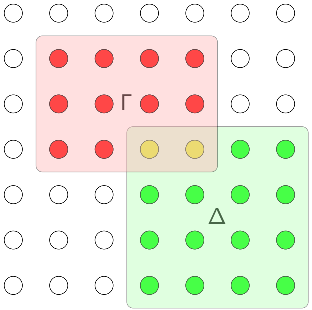

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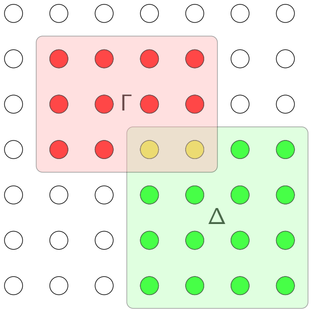

Lemma [Trivedi’03, Jiang’09,Fried et al’16]

Every Skolem function for y in φ must

- Evaluate to 1 in $(\Delta \setminus \Gamma)$ and to 0 in $(\Gamma \setminus \Delta)$
- Be an **interpolant** of $(\Delta \setminus \Gamma)$ and $(\Gamma \setminus \Delta)$

"Guess"-ing candidate Skolem functions ($|Y| = 1$)

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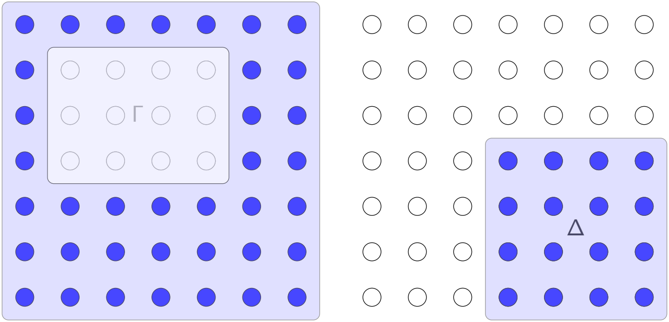

— Specific interpolants of $(\Delta \setminus \Gamma)$ & $(\Gamma \setminus \Delta)$

- $\neg\Gamma \triangleq \varphi(X, y)[y \mapsto 1] \equiv \varphi(X, 1)$
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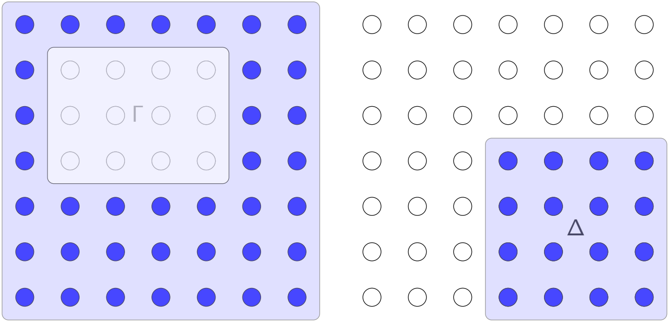

— Specific interpolants of $(\Delta \setminus \Gamma)$ & $(\Gamma \setminus \Delta)$

- $\neg\Gamma \triangleq \varphi(X, y)[y \mapsto 1] \equiv \varphi(X, 1)$: Easy solution for 1 output var
- $\Delta \triangleq \neg\varphi(X, y)[y \mapsto 0] \equiv \neg\varphi(X, 0)$.

“Guess”-ing Game: $(|\mathsf{Y}| = 2)$

Suppose relational spec is $\varphi(\mathsf{X}, y_1, y_2)$

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  - $F_2(\mathrm{X}) = \neg\Gamma^{y_2}(\mathrm{X}) = 0$

“Guess”-ing Game: ($|Y| > 2$)

Suppose relational spec is $\varphi(X, y_1, \boxed{Y_{2..m}})$

- Skolem function for $\boxed{Y_{2..m}}$ depends on that for y_1 in general
- For what values of X can we not set y_1 to 1 (or 0)?
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Drawbacks of approach:

- Existential quant elimination over long sequences of outputs expensive
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Can we work around these drawbacks?

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A $|X|$-input, $|Y|$-output circuit computing the desired Skolem function vector $(F_1, \ldots F_m)$ can be constructed with

- $\#\text{gates} \leq \sum_{i=1}^{m} \#\text{gates}(G_i) + 2m$
- $\#\text{wires} \leq \sum_{i=1}^{m} \#\text{wires}(G_i) + \frac{m(m-1)}{2}$

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Sufficient to compute the G_i functions

“Guess”-ing Compositionally: A High-level View

Suppose $\varphi(\mathsf{X}, Y) \equiv \varphi_1(\mathsf{X}, Y) \;\wedge\; \varphi_2(\mathsf{X}, Y)$, where $Y = y_1, \ldots y_m$

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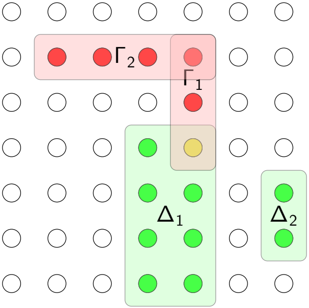

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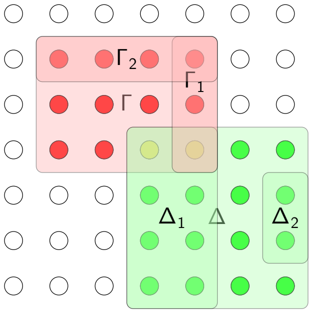

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If $\Gamma^{y_1} \triangleq \neg\exists y_2 \ldots y_m \ (\varphi_1 \wedge \varphi_2)(X, 1, \ldots)$, then $\Gamma_1^{y_1} \vee \Gamma_2^{y_1} \ \Rightarrow \Gamma^{y_1}$
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    - ${\Gamma_1}^{y_i} \vee (\wedge)\; {\Gamma_2}^{y_i} \Rightarrow (\Leftrightarrow)\Gamma^{y_i}$
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- Fortunately, non-trivial under-approx of Γ^{y_i} and Δ^{y_i} not hard to obtain

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- Similarly for Δ^{y_i}

"Check"-ing correctness of candidate Skolem func. vector

Given candidate Skolem functions $F_1, \ldots F_m$,

$$\text{Is } \forall \mathsf{X}(\ \exists \mathsf{Y} \varphi(\mathsf{X}, \mathsf{Y}) \ \Leftrightarrow \ \varphi(\mathsf{X}, \mathsf{F}(\mathsf{X})\ )\ ?$$

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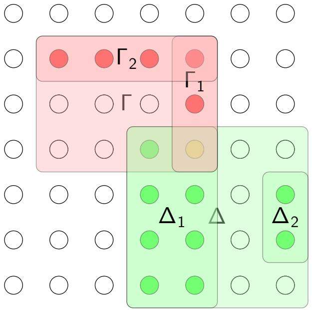
    - $\varphi(\sigma(X), \sigma(Y)) = 0$
    - $\varphi(\sigma(X), \sigma(Y')) = 1$
  - σ is counterexample to the claim that $F_1, \ldots F_m$ is a correct Skolem function vector

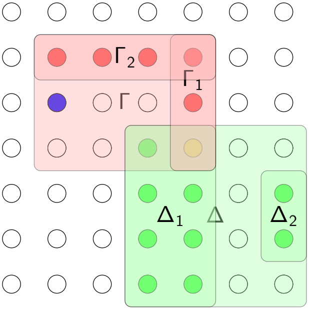
Repairing candidate Skolem functions: A High-level View

$$\varphi(X, Y) \equiv \varphi_1(X, Y) \ \wedge \ \varphi_2(X, Y)$$



$$\varphi(X, Y) \equiv \varphi_1(X, Y) \;\wedge\; \varphi_2(X, Y)$$

Counterexample



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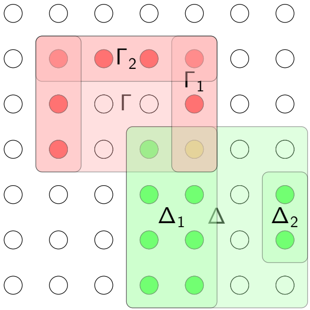
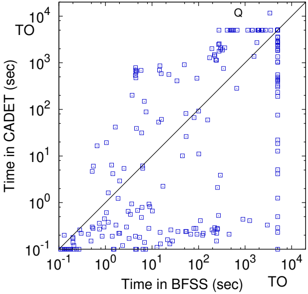
Expansion around CEX

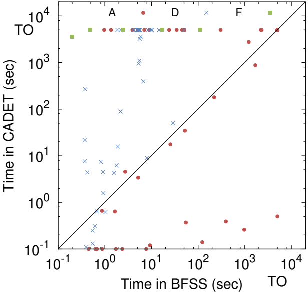


- Always work with under-approximations of Γ and Δ
- Since "proposed" Skolem function is $\neg\Gamma$, intermediate approximations of Skolem functions are over-approximations (abstractions)

BFSS vis-a-vis CADET [Rabe & Seshia'16]

[Comparisons with other tools in paper]





Q: QBFEval, A: Arithmetic, F: Factorization, D: Disjunctive Decomposition. TO: Timeout (3600 sec)

Comparison with other tools

BFSS vis-a-vis CADET [Rabe & Seshia'16]
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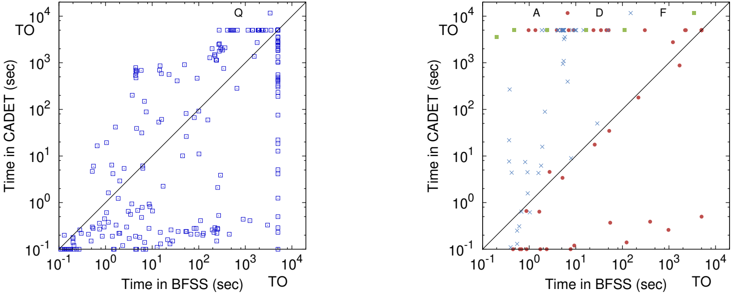

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- Mixed results: tools have orthogonal strengths
- Using CADET and BFSS as a portfolio solver sounds promising