Quick recap: Game Theory
  * Analytical approach for predicting reasonable outcome
  * Building blocks: players, strategies, utilities
  * Difference between action and strategy
  * Key assumptions: rationality and intelligence

Example to illustrate: Game of Chess (von Neumann and Morgenstern, 1944)

Formal description
  * Two player game: White and Black - 16 pieces each.
  * Every piece has some legal moves - ACTIONS
  * Starts with W, players take turns
  * Ends: W win, if W captures B king
        B win, if B captures W king
        Draw, if nobody has legal moves but kings are not in check, both players agree to a draw, board position is such that nobody can win, many more ...

Natural questions from a theorist’s perspective
  * Does W have a winning strategy, i.e., a plan of moves s.t. it wins IRRESPECTIVE of the moves of B?
  * Does B have a winning strategy?
  * Or at least guarantee a draw?
  * Neither may be possible — not synonymous with end of game.
What is a strategy?

In the context of chess, board position if different from Game Situation more than one sequence of moves can bring to the same board position.

denote a board position by $x_k$

Game Situation is a finite sequence $(x_0, x_1, x_2, \ldots, x_k)$ of board positions s.t.
- $x_0$ is the opening board position
- $x_k \rightarrow x_{k+1}$, k even - created by a single action of W
- k odd - created by a single action of B

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**Board positions**

- $x_0$
- $x_1$
- $x_2$

**W moves (actions)**

**B moves**

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Board positions may repeat in this tree, but a vertex is unique — Game Situation

**strategy**: mapping from game situation to action, i.e. what action to take at every vertex of this game tree. A complete contingency plan.
A strategy for \( W \) is a function \( S_W \) that associates every game situation \((x_0, x_1, \ldots, x_k) \in \mathcal{H} \) (set of all game situations), \( k \) even, with a board position \( x_{k+1} \) such that the move \( x_k \rightarrow x_{k+1} \) is a single valid move of \( W \).

Similar definition of \( S_B \) for \( B \).

Note:
- strategy pair \((S_W, S_B)\) determines an outcome also called one play of the game – a path through the game tree

Questions:
1. this is a finite game – where does it end?
2. can a player guarantee an outcome?

The game ends: \( \circ \) \( W \) wins or \( \bullet \) \( B \) wins or \( \odot \) Draw.

A winning strategy for \( W \) is a strategy \( S_W^* \) s.t. for every \( S_B \) \((S_W^*, S_B)\) ends in a win for \( W \).

A strategy guaranteeing at least a draw for \( W \) is \( S_W' \) s.t. for every \( S_B \), \((S_W', S_B)\) either ends in a draw or win for \( W \).

Analogous definitions of \( S_B^* \) and \( S_B' \)

Not obvious if such strategies exist
An early result of Game Theory (von Neumann, 1928)

In chess, one and only one of the following statements is true:

1. W has a winning strategy
2. B has a winning strategy
3. Each player has a strategy guaranteeing a draw

- there were other possibilities, e.g., nothing can be guaranteed
- it does not say what is that strategy
  actually it is not known: which one is true and what is that strategy

Chess would have been a boring game if any of these answers were known.