An early result of Game Theory (von Neumann, 1928)

In chess, one and only one of the following statements is true:

1. W has a winning strategy
2. B has a winning strategy
3. Each player has a strategy guaranteeing a draw

Proof: Each vertex is a game situation

\( \Gamma(x) \): subtree rooted at x (includes itself)

\( n_x \): number of vertices in \( \Gamma(x) \)

\( y \) is a vertex in \( \Gamma(x) \), \( y \neq x \)

\( \Gamma(y) \) is a subtree of \( \Gamma(x) \), \( n_y < n_x \)

\( n_x = 1 \Rightarrow x \) is a terminal vertex

The proof is via induction on \( n_x \)

The theorem holds for \( n_x = 1 \), why?

- if W king is removed, B wins
- if B king is removed, W wins
- if both kings present, but game ends — draw

Suppose \( x \) is a vertex with \( n_x > 1 \)

Induction hypothesis: for all vertices \( y \in \Gamma(y) \), s.t. \( n_y < n_x \), in particular, \( \Gamma(y) \) is a subgame of \( \Gamma(x) \)

The statement holds
\[ C(x) = \text{Vertices reachable from } x \text{ in one step.} \]

Case (i) if \( \exists y_0 \in C(x), \text{ s.t. } (1) \text{ is true in } \Gamma(y_0), \text{ then } (1) \text{ is true in } \Gamma(x) \)

W just picks that

Case (ii) if \( \forall y \in C(x), \text{ (2) is true, then (2) is true in } \Gamma(x) \)

B sees that action and picks the appropriate action to win.

Case (iii)

- (i) does not hold, W does not have a winning strategy in any \( y \in C(x) \)
  
  Since induction hypothesis holds for every \( y \in C(x) \), either B has a winning strategy or both have draw-guaranteeing strategy

- (ii) doesn’t hold, \( \exists y' \in C(x) \) where B doesn’t have a winning strategy
  
  Since (i) doesn’t hold either, W can’t guarantee a win in \( y' \)
  
  Hence they both have strategies guaranteeing a draw.

W picks the action to reach \( y' \).

B picks action that guarantees a draw or win.

This concludes the proof.

Exercise: prove this when the length of game is infinite, \( (\exists x \land \exists y \land \exists z) \)

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