Matrix games (Two player zero sum games)

A special class with certain nice properties of the stability and security notions

$\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$, with $N = \{1, 2\}$, $u_1 + u_2 \equiv 0$

Example: Penalty shootout game

G	L	R
L	−1, 1	1, −1
R	1, −1	−1, 1

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Possible to represent the game with one matrix u, considering the utilities of only player 1

An arbitrary game

	L	C	R
T	3, −3	−5, 5	−2, 2
M	1, −1	4, −4	1, −1
B	6, −6	−3, 3	−5, 5

Player 2's utilities are negative of the matrix

Player 2's maxmin strategies are the minmax of this matrix (security criterion)

u	L	R	maxmin
L	−1	1	−1
R	1	−1	−1
minmax	1	1	

u	L	C	R	maxmin
T	3	−5	−2	−5
M	1	4	1	1
B	6	−3	−5	−5
minmax	6	4	1	

What are the PSNEs of these games?

Saddle point (of a matrix): The value is maximum for player 1, minimum for player 2.

Rephrase: what are the saddle point of the two games?

Theorem: In a matrix game with utility matrix u, (s_1^*, s_2^*) is a saddle point if and only if it is a PSNE.

Proof: (s_1^*, s_2^*) is a saddle point $\Leftrightarrow$

$$u(s_1^*, s_2^*) \geq u(s_1, s_2^*), \forall s_1 \in S_1, \text{ and } u(s_1^*, s_2^*) \leq u(s_1^*, s_2) \quad \forall s_2 \in S_2$$

$\Leftrightarrow$ it is a PSNE, since $u_1 \equiv u$, $u_2 = -u$.

Consider the maxmin and minmax values

$$\left.\begin{aligned} \underline{v} &= \max_{s_1 \in S_1} \min_{s_2 \in S_2} u(s_1, s_2) \\ \overline{v} &= \min_{s_2 \in S_2} \max_{s_1 \in S_1} u(s_1, s_2) \end{aligned}\right\} \text{How are they related?}$$

Lemma: For matrix games $\overline{v} \geq \underline{v}$.

Proof: $$u(s_1, s_2) \geq \min_{t_2 \in S_2} u(s_1, t_2)$$

$$\max_{t_1 \in S_1} u(t_1, s_2) \geq \max_{t_1 \in S_1} \min_{t_2 \in S_2} u(t_1, t_2) \quad \forall s_2$$

$$\min_{t_2 \in S_2} \max_{t_1 \in S_1} u(t_1, t_2) \geq \max_{t_1 \in S_1} \min_{t_2 \in S_2} u(t_1, t_2) \qquad \square$$