Earlier matrix game examples

\[
\begin{array}{ccc|c}
U & L & R & \text{max-min} \\
\hline
L & -1 & 1 & -1 \\
R & 1 & -1 & -1 \\
\hline
\text{min-max} & 1 & 1 & 1
\end{array}
\]

\[
\begin{array}{cccc|c}
U & L & C & R & \text{max-min} \\
\hline
T & 3 & -5 & 2 & -5 \\
M & 1 & 4 & 1 & 1 \\
B & 6 & -3 & -5 & -5 \\
\hline
\text{min-max} & 6 & 4 & 1 & 1
\end{array}
\]

\[
\bar{v} = 1 > -1 = \underline{v} \\
\text{PSNE doesn't exist}
\]

\[
\bar{v} = 1 = \underline{v} \\
\text{PSNE exists}
\]

Define \( s_1^* \in \arg \max \min_{\sigma_1 \in S_1} \min_{\sigma_2 \in S_2} U(\sigma_1, \sigma_2) \): max-min strategy of 1

\[
s_2^* \in \arg \min \max_{\sigma_2 \in S_2} \max_{\sigma_1 \in S_1} U(\sigma_1, \sigma_2) \): min-max strategy of 2

Theorem: A matrix game \( U \) has a PSNE (saddle point) if and only if \( \bar{v} = \underline{v} = U(s_1^*, s_2^*) \), where \( s_1^* \) and \( s_2^* \) are max-min and min-max strategies for players 1 and 2 respectively. In particular, \( (s_1^*, s_2^*) \) is a PSNE.

Proof: \((\Rightarrow)\) i.e., PSNE \( \Rightarrow \bar{v} = \underline{v} = U(s_1^*, s_2^*)

Say the PSNE is \( (s_1^*, s_2^*) \), i.e., \( U(s_1^*, s_2^*) \geq U(s_1, s_2^*) \), \( \forall s_1 \in S_1 \)

\[
\Rightarrow U(s_1^*, s_2^*) \geq \max_{t_1 \in S_1} U(t_1, s_2^*)
\]

\[
\Rightarrow \min_{t_2 \in S_2} \max_{t_1 \in S_1} U(t_1, t_2), \text{ since } s_2^* \text{ is a specific strategy}
\]

\[
= \bar{v}
\]
Similarly, using the same argument for player 2, we get
\( \bar{v} \geq u(A_1^*, A_2^*) \), for player 2 utility \( u_2 = -u \)

But \( \bar{v} \geq v \) [from previous lemma]

Hence, \( u(A_1^*, A_2^*) \geq \bar{v} \geq v \geq u(A_1^*, A_2^*) \)

\( \Rightarrow u(A_1^*, A_2^*) = \bar{v} = v \), also implies that the maximin for 1
and minimax for 2 are \( A_1^* \) and \( A_2^* \) resp.

\((\Leftarrow)\) given \( u(A_1^*, A_2^*) = \bar{v} = v \), \( A_1^*, A_2^* \) are maximin and minimax
\( = v \) (say) resp. for 1 and 2.

\[ u(A_1^*, A_2^*) \geq \min_{t_2 \in S_2} u(A_1^*, t_2) \quad \text{by defn of min} \]

\[ \forall A_2 \in S_2 \]

\[ = \max_{t_1 \in S_1} \min_{t_2 \in S_2} u(t_1, t_2) \quad \text{since \( A_1^* \) is the maximin strategy for 1.} \]

\[ = v \quad \text{given} \]

Similarly, show \( u(A_1, A_2^*) \leq v \quad \forall A_1 \in S_1 \)

but \( v = u(A_1^*, A_2^*) \). Substitute and get that \((A_1^*, A_2^*)\) is a PSNE