**MSNE characterization theorem**

**Theorem:** A mixed strategy profile \((\sigma_i^*, \sigma_i^*)\) is a MSNE iff \(\forall i \in N\)

1. \(u_i(\sigma_i, \sigma_i^*)\) is the same for all \(\sigma_i \in \delta(\sigma_i^*)\)
2. \(u_i(\sigma_i, \sigma_i^*) \geq u_i(\sigma_i', \sigma_i^*), \forall \sigma_i' \in \delta(\sigma_i^*), \sigma_i' \notin \delta(\sigma_i^*)\)

**Observations:**

1. \[\max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_i^*) = \max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_i^*)\]
   maximizing w.r.t. a distribution \(\equiv\) whole probability mass at max

2. \[\max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_i^*) \geq \max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_i^*) = \max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_i^*)\]
   the maximizer must lie in \(\delta(\sigma_i^*)\) - if no maximizer in \(\delta(\sigma_i^*)\)
   then put all probability mass on that \(\sigma_i \notin \delta(\sigma_i^*)\) that has the maximum value of the utility - \((\sigma_i^*, \sigma_i^*)\) is not a MSNE.

**Proof:** (\(\Rightarrow\)) given \((\sigma_i^*, \sigma_i^*)\) is an MSNE

\[u_i(\sigma_i, \sigma_i^*) = \max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_i^*)\]
\[= \max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_i^*)\]
\[= \max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_i^*)\]
\[= \max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_i^*)\] ---- (1)
by definition of expected utility
\[
U_i(\sigma_i^*, \sigma_{-i}^*) = \sum_{\lambda_i \in \mathcal{S}_i} \sigma_i^*(\lambda_i) U_i(\lambda_i, \sigma_{-i}^*)
\]
\[
= \sum_{\lambda_i \in \mathcal{S}(\sigma_i^*)} \sigma_i^*(\lambda_i) U_i(\lambda_i, \sigma_{-i}^*)
\]  
\[\text{positive}\]
\[\text{positive}\]

(1) and (2) are equal — max is equal to weighted average

— can happen only when all values are same. proves condition 1

for condition 2: suppose for contradiction

\[\exists \lambda_i \in \mathcal{S}(\sigma_i^*) \text{ and } \lambda_i' \notin \mathcal{S}(\sigma_i^*)\]

s.t. \[U_i(\lambda_i, \sigma_{-i}^*) < U_i(\lambda_i', \sigma_{-i}^*)\]

shift the probability mass \[\tau_i^*(\lambda_i')\] to \[\lambda_i'\], this new mixed strategy gives a strict better utility — contradiction to MSNE.

(\Rightarrow) Given the two conditions of the theorem hold

let \[U_i(\lambda_i, \sigma_{-i}^*) = m_i(\sigma_{-i}^*), \forall \lambda_i \in \mathcal{S}(\sigma_{-i}^*)\] — condition 1

note \[m_i(\sigma_{-i}^*) = \max_{\lambda_i \in \mathcal{S}_i} U_i(\lambda_i, \sigma_{-i}^*)\] — condition 2

\[
U_i(\sigma_i^*, \sigma_{-i}^*) = \sum_{\lambda_i \in \mathcal{S}(\sigma_i^*)} \tau_i^*(\lambda_i) U_i(\lambda_i, \sigma_{-i}^*)
\]

= \[
m_i(\sigma_{-i}^*)
\] — previous conclusion

= \[
\max_{\lambda_i \in \mathcal{S}_i} U_i(\lambda_i, \sigma_{-i}^*)
\] — previous conclusion

\[
\max_{\sigma_i \in \Delta(\mathcal{S}_i)} U_i(\sigma_i, \sigma_{-i}^*) \geq U_i(\sigma_i^*, \sigma_{-i}^*), \forall \sigma_i \in \Delta(\mathcal{S}_i)
\]

from the observation

algorithmic way to find MSNE