MSNE characterization Theorem to algorithm.

**NFG** $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$

All possible supports of $S_1 \times S_2 \times \ldots \times S_n$

number $= K = (2^{|S_1|} - 1) \times (2^{|S_2|} - 1) \times \ldots \times (2^{|S_n|} - 1)$

for every support profile $X_1 \times X_2 \ldots \times X_n$, where $X_i \subseteq S_i$

solve the following feasibility program

$$w_i = \sum_{A_i \in S_i} \left( \prod_{j \neq i} \sigma_j(A_j) \right) u_i(A_i, A_i), \forall A_i \in X_i, \forall i \in N$$  \text{--- Cond 1}

$$w_i \geq \sum_{A_i \in S_i} \left( \prod_{j \neq i} \sigma_j(A_j) \right) u_i(A_i, A_i), \forall A_i \in S_i \setminus X_i, \forall i \in N$$  \text{--- Cond 2}

$$\sigma_j(A_j) > 0, \forall A_j \in S_j, \forall j \in N, \text{ and } \sum_{A_j \in S_j} \sigma_j(A_j) = 1, \forall j \in N.$$

feasibility program with variables $w_i, i \in N, \sigma_j(A_j), A_j \in S_j, j \in N$.

Remarks: this is not a linear program unless $n=2$

For general games, there is no poly-time algorithm.

Problem of finding an MSNE is PPAD-complete [Polynomial Parity Argument on Directed graphs].

MSNE and dominance

The previous algorithm can be applied to a smaller set of strategies by removing the dominated strategies.

Dominated strategy in this game?

Domination can also be via mixed strategy

Weak dominated strategy removal can remove equilibrium

for strictly dominated strategies

Theorem: If a pure strategy $s_i$ is strictly dominated by a mixed strategy $\sigma_i \in \Delta(s_i)$, then in every MSNE of the game, $s_i$ is chosen with probability zero.

So, can remove without loss of equilibrium.

Existence of MSNE

Finite game: number of players and the strategies are finite

Theorem (Nash 1951)

Every finite game has a (mixed) Nash equilibrium.
Proof requires a few tools and a result from real analysis

- A set $S \subseteq \mathbb{R}^n$ is convex if $\forall x, y \in S$ and $\forall \lambda \in [0, 1]$, $\lambda x + (1-\lambda) y \in S$

- A set $S \subseteq \mathbb{R}^n$ is closed if it contains all its limit points (points whose every neighborhood contains a point in $S$ - a set not closed $[0, 1)$ - every ball of radius $\epsilon > 0$ around 1 has a member of $[0, 1)$, but 1 is not in the set $[0, 1)$)

- A set $S \subseteq \mathbb{R}^n$ is bounded if $\exists x_0 \in \mathbb{R}^n$ and $R \in (0, \infty)$ s.t.
  $\forall x \in S$, $\|x - x_0\|_2 < R$

- A set $S \subseteq \mathbb{R}^n$ is compact if it is closed and bounded.

A result from real analysis (without proof)

**Brouwer's fixed point theorem**

If $S \subseteq \mathbb{R}^n$ is convex and compact and $T : S \rightarrow S$, is continuous
Then $T$ has a fixed point, i.e., $\exists x^* \in S$ s.t. $T(x^*) = x^*$.