Equilibrium notions in NIEFGs

- Can extend the subgame perfection of PIEFG, but since the nodes/histories are uncertain, we need to extend to mixed strategies.

- Because of the information sets, best response cannot be defined without the belief of each player.

Belief is a conditional probability distribution over the histories in an information set - conditioned on reaching the information set.

Example: an NIEFG with perfect recall, i.e., mixed and behavioral strategies are equiv.

\[
\begin{align*}
\text{Ex 7.38 (MSZ)} & \quad \frac{5}{12} \quad \frac{4}{12} \quad \frac{3}{12} \\
\ell & \quad x_2 & \quad L & \quad x_1 & \quad I_1^1 & \quad L & \quad x_3 & \quad I_2^1 & \quad R & \quad 2, 0 \\
2, 1 & \quad L_1 & \quad I_2^2 & \quad x_4 & \quad x_5 & \quad I_1^2 & \quad R_1 & \quad L_1 & \quad 0, 0 & \quad 0, -1 & \quad 5, 3 \\
0, 2 & \quad 3, 0 & \quad 2, 2 & \quad 0, 3 & \quad & \quad & \quad & \quad & \\
& \quad R_1 & \quad L_2 & \quad R_1 & \quad L_2 & \quad R_2 & \quad L_2 & \quad & \quad & \quad & \\
& \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \\
& \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \\
\end{align*}
\]

Consider the behavioral strategy profile: \( \sigma_1 \) at \( I_1^1 \) \( \left( \frac{5}{12} L, \frac{4}{12} M, \frac{3}{12} R \right) \)

at \( I_1^2 \), choose \( R_1 \), at \( I_1^3 \), choose \( L_2 \)

\( \sigma_2 \) : choose \( L \)

Q: Is this an equilibrium? Which implies

- are the Bayesian beliefs consistent with \( P_\theta \) - that visits vertex \( x \) w.p. \( P_\theta(x) \)

- The actions and beliefs are consistent for every player, i.e., maximizes their expected utility.
- Player 1, at $I_1^3$, believes that $\tau_6$ is reached w.p. 1
  
  if the belief was $> \frac{2}{7}$ in favor of $\tau_7$, should have chosen $R_2$
  
  choose an action maximizing expected utility at each information set
  
  - sequential rationality

  The strategy vector $\sigma$ induces the following probabilities to the vertices

  
  $P_\sigma(\tau_2) = \frac{5}{12}$, $P_\sigma(\tau_3) = \frac{4}{12}$, $P_\sigma(\tau_4) = P(\tau_5) = P_\sigma(\tau_7) = 0$, $P_\sigma(\tau_8) = \frac{4}{12}$

  - Player 2, at $I_2^1$, believes that $\tau_3$ is reached w.p.

  
  $P_\sigma(\tau_3 | I_2^1) = \frac{P_\sigma(\tau_3)}{P_\sigma(\tau_2) + P_\sigma(\tau_3)} = \frac{4/12}{4/12 + 5/12} = \frac{4}{9}$

  Similarly, $P_\sigma(\tau_2 | I_2^1) = \frac{5}{9}$

  Is the action of player 2 sequentially rational w.r.t her belief?

  by picking $L$, her expected utility $= \frac{5}{9} \times 1 + \frac{4}{9} \times 2 = \frac{13}{9}$, this is larger
  than any other choice of actions.

  - Given this, what will be the sequentially rational strategy of player 1 at $I_1^1$?

  - L, M, R all gives the same expected utility for 1 (utility = 2)

  mixed/behavioral strategy profile $\sigma$ is sequentially rational for all players
Formal definitions

1. **Belief**: Let the information sets of player $i$ be \( I_i = \{ I_i^1, \ldots, I_i^{k(i)} \} \). The belief of player $i$ is a mapping $\mu^j_i : I_i^j \to [0,1]$, s.t.

\[
\sum_{x \in I_i^j} \mu^j_i(x) = 1
\]

2. **Bayesian belief**: A belief $\mu_i = (\mu^j_i, j = 1, \ldots, k(i))$ of player $i$ is Bayesian with the behavioral strategy $\sigma$, if it is derived from $\sigma$ using Bayes' rule, i.e.,

\[
\mu^j_i(x) = \frac{P_\sigma(x)}{\sum_{y \in I_i^j} P_\sigma(y)}, \quad \forall x \in I_i^j, \forall j = 1, 2, \ldots, k(i)
\]

3. **Sequential rationality**: A strategy $\sigma_i$ of player $i$ at an information set $I_i^j$ is sequentially rational given $\sigma_i$ and partial belief $\mu^j_i$ if

\[
\sum_{x \in I_i^j} \mu^j_i(x) u_i(\sigma_i, \sigma_{-i} | x) \geq \sum_{x \in I_i^j} \mu^j_i(x) u_i(\sigma_i, \sigma_{-i} | x)
\]

The tuple $(\sigma, \mu)$ is sequentially rational if it is sequentially rational for every player at every information set.

The tuple $(\sigma, \mu)$ is also called an assessment.

Sequential rationality is a refinement of Nash equilibrium.
The notion coincides with SPNE when applied to PIEFGs.

**Theorem:** In a PIEFG, a behavioral strategy profile \( \sigma \) is an SPNE iff the tuple \( (\sigma, \mu) \) is sequentially rational.

[In PIEFG, every information set is a singleton; \( \mu \) is the degenerate distribution at that singleton.]

**Equilibrium with sequential rationality**

**Perfect Bayesian equilibrium**

An assessment \( (\sigma, \mu) \) is a perfect Bayesian equilibrium (PBE) if for every player \( i \in N \)

1. \( \mu_i \) is Bayesian w.r.t. \( \sigma \), and
2. \( \sigma_i \) is sequentially rational given \( \sigma_i \) and \( \mu_i \).

Often represented only with \( \sigma \), since \( \mu \) is obtained from \( \sigma \).

Self-enforcing (like the SPNE) in a Bayesian way.