## Bayesian games

$$\langle N, (\Theta_i)_{i\in N}, P, (\Gamma_\theta)_{\theta \in \times_{i\in N} \Theta_i} \rangle$$

$$\Gamma_\theta = \langle N, (A_i(\theta))_{i\in N}, (u_i(\theta))_{i\in N} \rangle$$ [we assume $A_i(\theta) = A_i, \forall \theta$]

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Strategy : a plan to map type to action

Pure : $s_i : \Theta_i \to A_i$ , Mixed : $\sigma_i : \Theta_i \to \Delta A_i$

The player can experience its utility in two stages (depending on the realization of $\theta_i$).

Ex-ante utility : expected utility before observing own type

$$U_i(\sigma) = \sum_{\theta \in \Theta} P(\theta)\, u_i(\sigma(\theta); \theta)$$

$$= \sum_{\theta \in \Theta} P(\theta) \sum_{(a_1,\dots,a_n)\in A} \prod_{j \in N} \sigma_j(\theta_j)[a_j]\, u_i(a_1, \dots, a_n; \theta_1, \dots, \theta_n)$$

The belief of player $i$ over others' types changes after observing her own type $\theta_i$ according to Bayes rule on $P$

$$P(\underline{\theta}_{-i} \mid \theta_i) = \frac{P(\theta_i, \underline{\theta}_{-i})}{\sum_{\underline{\tilde{\theta}}_{-i} \in \Theta_{-i}} P(\theta_i, \underline{\tilde{\theta}}_{-i})}$$ ; positive marginals assumption is crucial

Ex-interim utility : expected utility after observing one's own type

$$U_i(\sigma \mid \theta_i) = \sum_{\underline{\theta}_{-i} \in \Theta_{-i}} P(\underline{\theta}_{-i} \mid \theta_i)\, u_i(\sigma(\theta); \theta)$$

Special case : for independent types, observing $\theta_i$ does not give any information on $\underline{\theta}_{-i}$. Both utilities are same.

Relationship between these two utilities

$$u_i(\sigma) = \sum_{\theta_i \in \Theta_i} P(\theta_i)\, u_i(\sigma \mid \theta_i)$$

Example 1: Two player bargaining game

Player 1: seller, type: price at which he is willing to sell

Player 2: buyer, type: price at which he is willing to buy

$\Theta_1 = \Theta_2 = \{1, 2, \ldots, 100\}$, $A_1 = A_2 = \{1, 2, \ldots, 100\}$, bids

If the bid of the seller is smaller or equal to that of the buyer, trade happens at a price average of the two bids. Else, trade does not happen.

Suppose type generation is independent and uniform over $\Theta_1, \Theta_2$ resp.

$$P(\theta_2 \mid \theta_1) = P(\theta_2) = \frac{1}{100} \quad \forall \theta_1, \theta_2.$$

$$P(\theta_1 \mid \theta_2) = P(\theta_1) = \frac{1}{100}, \quad \forall \theta_1, \theta_2.$$

$$u_1(a_1, a_2; \theta_1, \theta_2) = \begin{cases} \frac{a_1 + a_2}{2} - \theta_1 & \text{if } a_2 \geq a_1 \\ 0 & \text{ow} \end{cases}$$

$$u_2(a_1, a_2; \theta_1, \theta_2) = \begin{cases} \theta_2 - \frac{a_1 + a_2}{2} & \text{if } a_2 \geq a_1 \\ 0 & \text{ow} \end{cases}$$

Common prior $P(\theta_1, \theta_2) = \frac{1}{10000}$, $\forall \theta_1, \theta_2$

Example 2: Sealed bid auction

Two players, both willing to buy an object. Their values and bids lie in $[0,1]$

allocation function: $O_1(b_1,b_2) = \begin{cases} 1 & \text{if } b_1 \geq b_2 \\ 0 & \text{ow} \end{cases}$ | $O_2(b_1,b_2) = \begin{cases} 1 & \text{if } b_2 > b_1 \\ 0 & \text{ow} \end{cases}$

beliefs: $f(\theta_2|\theta_1) = 1, \ \forall \theta_1, \theta_2$

$f(\theta_1|\theta_2) = 1, \ \forall \theta_1, \theta_2$

$f(\theta_1, \theta_2) = 1, \ \forall (\theta_1, \theta_2) \in [0,1]^2$

$u_i(b_1, b_2; \theta_1, \theta_2) = O_i(b_1,b_2)(\theta_i - b_i)$ [winner pays his bid]