Bayesian games $\langle N, (\Theta_i)_{i \in N}, P, (\Gamma_\theta)_{\theta \in \prod_{i \in N} \Theta_i} \rangle$

$\Gamma_\theta = \langle N, (A_i(\theta))_{i \in N}, (U_i(\theta))_{i \in N} \rangle$ [we assume $A_i(\theta) = A_i, \forall \theta$]

Strategy: a plan to map type to action

Pure: $S_i : \Theta_i \rightarrow A_i$, Mixed: $\sigma_i : \Theta_i \rightarrow \Delta A_i$

The player can experience its utility in two stages (depending on the realization of $\Theta_i$).

Ex-ante utility: expected utility before observing own type

$U_i(\sigma) = \sum_{\theta \in \Theta} P(\theta) U_i(\sigma(\theta) ; \theta)$

$= \sum_{\theta \in \Theta} P(\theta) \sum_{(a_i, \ldots, a_n) \in A_i} \prod_{j \in N} \sigma_j(\theta_j) [a_j] U_i(a_1, \ldots, a_n; \theta_1, \ldots, \theta_n)$

The belief of player $i$ over others’ types changes after observing her own type $\Theta_i$ according to Bayes rule on $P$

$P(\theta_i | \Theta_i) = \frac{P(\theta_i, \Theta_i)}{\sum_{\tilde{\Theta}_i \in \Theta_i} P(\tilde{\Theta}_i, \Theta_i)}$; positive marginals assumption is crucial

Ex- interim utility: expected utility after observing one’s own type

$U_i(\sigma | \Theta_i) = \sum_{\theta_i \in \Theta_i} P(\theta_i | \Theta_i) U_i(\sigma(\theta) ; \theta)$

Special case: for independent types, observing $\Theta_i$ does not give any information on $\Theta_i$. Both utilities are same.
Relationship between these two utilities

\[ U_i(\sigma) = \sum_{\theta_i \in \Theta_i} P(\theta_i) \cdot U_i(\sigma | \theta_i) \]

**Example 1:** Two player bargaining game

**Player 1:** seller, type: price at which he is willing to sell

**Player 2:** buyer, type: price at which he is willing to buy

\[ \Theta_1 = \Theta_2 = \{1, 2, \ldots, 100\}, \quad A_1 = A_2 = \{1, 2, \ldots, 100\}, \text{ bids} \]

If the bid of the seller is smaller or equal to that of the buyer, trade happens at a price average of the two bids. Else, trade does not happen.

Suppose type generation is independent and uniform over \( \Theta_1, \Theta_2 \) resp.

\[ P(\theta_2 | \theta_1) = P(\theta_2) = \frac{1}{100}, \quad \forall \theta_1, \theta_2. \]

\[ P(\theta_1 | \theta_2) = P(\theta_1) = \frac{1}{100}, \quad \forall \theta_1, \theta_2. \]

\[ U_1(a_1, a_2; \theta_1, \theta_2) = \begin{cases} \frac{a_1 + a_2}{2} - \theta_1 & \text{if } a_2 \geq a_1 \\ 0 & \text{otherwise} \end{cases} \]

\[ U_2(a_1, a_2; \theta_1, \theta_2) = \begin{cases} \theta_2 - \frac{a_1 + a_2}{2} & \text{if } a_2 \geq a_1 \\ 0 & \text{otherwise} \end{cases} \]

**Common prior** \( P(\theta_1, \theta_2) = \frac{1}{10000}, \quad \forall \theta_1, \theta_2 \)
Example 2: Sealed bid auction

Two players, both willing to buy an object. Their values and bids lie in \([0, 1]\).

Allocation function:
\[
O_1(b_1, b_2) = \begin{cases} 
1 & \text{if } b_1 \geq b_2 \\
0 & \text{otherwise} 
\end{cases} 
\]
\[
O_2(b_1, b_2) = \begin{cases} 
1 & \text{if } b_2 \geq b_1 \\
0 & \text{otherwise} 
\end{cases} 
\]

Beliefs:
\[
f(\theta_2 | \theta_1) = 1, \quad \forall \theta_1, \theta_2 
\]
\[
f(\theta_1 | \theta_2) = 1, \quad \forall \theta_1, \theta_2 
\]
\[
f(\theta_1, \theta_2) = 1, \quad \forall \theta_1, \theta_2 
\]
\[
f(\theta_1, \theta_2) \in [0, 1]^2 
\]

\[
U_i(b_1, b_2; \theta_1, \theta_2) = O_i(b_1, b_2)(\theta_i - b_i) \quad [\text{winner pays his bid}] 
\]