Bayesian equilibria in Bayesian games

Sealed bid auction

Two players, both willing to buy an object. Their values and bids lie in \([0,1]\)

allocation function: \(O_1(b_1,b_2) = I\{b_1 > b_2\}\); \(O_2(b_1,b_2) = I\{b_2 > b_1\}\)

beliefs: \(f_1(\theta_2|\theta_1) = 1, \forall \theta_1, \theta_2\)
\(f_2(\theta_1, \theta_2) = 1, \forall (\theta_1, \theta_2) \in [0,1]^2\)

(1) First price auction: if \(b_1 > b_2\), player 1 wins and pays her bid
on, player 2 wins and pays her bid

\[U_1(b_1, b_2, \theta_1, \theta_2) = (\theta_1 - b_1) I\{b_1 > b_2\}\]
\[U_2(b_1, b_2, \theta_1, \theta_2) = (\theta_2 - b_2) I\{b_1 < b_2\}\]

\(b_1 = \lambda_1(\theta_1), b_2 = \lambda_2(\theta_2), \) assume \(\lambda_i(\theta_i) = \alpha_i \theta_i, \) \(\alpha_i > 0, i = 1,2\).

To find The BE, we need to find \(\lambda_i^*(\theta_i)\) (on \(\alpha_i^*\)) that maximizes the ex-ante utility of player \(i\)

\[\max_{\sigma_i} U_i(\sigma_i, \sigma_i^*|\theta_i)\]

For player 1, this reduces to:

\[
\max_{b_1 \in [0, \alpha_2]} \int f(\theta_2|\theta_1)(\theta_1 - b_1) I\{b_1 > \frac{\alpha_1 \theta_2}{\alpha_2}\} d\theta_2
\]

\[
= \max_{b_1 \in [0, \alpha_2]} (\theta_1 - b_1) \frac{b_1}{\alpha_2} \Rightarrow b_1^* = \begin{cases} \frac{\theta_1}{2} & \text{if } \alpha_2 > \frac{\theta_1}{2} \\ \alpha_2 & \text{otherwise} \end{cases}
\]

\(\lambda_1^*(\theta_1) = \min \left\{ \frac{\theta_1}{2}, \alpha_2 \right\}, \lambda_2^*(\theta_2) = \min \left\{ \frac{\theta_2}{2}, \alpha_1 \right\}\)
If $\alpha_1 = \alpha_2 = \frac{1}{2}$, then $(\frac{\theta_1}{2}, \frac{\theta_2}{2})$ is a BE.

In the Bayesian game induced by uniform prior on first-price auction, bidding half the true value is a Bayesian equilibrium.

2. Second price auction: highest bidder wins but pays the second highest bid.

$U_1(b_1, b_2, \theta_1, \theta_2) = (\theta_1 - b_2) I\{b_1 > b_2\}$

$U_2(b_1, b_2, \theta_1, \theta_2) = (\theta_2 - b_1) I\{b_1 < b_2\}$

Player 1's bidding problem is to maximize

$$\int_0^1 f(\theta_2 | \theta_1) (\theta_1 - \alpha_2 \theta_2) I(\theta_2 \geq \frac{b_1}{\alpha_2}) \, d\theta_2$$

$$= \int_0^1 (\theta_1 - \alpha_2 \theta_2) I(\theta_2 \leq \frac{b_1}{\alpha_2}) \, d\theta_2$$

$$= \frac{1}{\alpha_2} (b_1, \theta_1 - \frac{\theta_2^2}{2}) \Rightarrow \text{maximized when } b_1 = \theta_1$$

Similarly for $b_2 = \theta_2$.

If the distributions of $\theta_1$ and $\theta_2$ were arbitrary but independent, the maximization problem would have been

$$\int_0^{b_1/\alpha_2} f(\theta_2) (\theta_1 - \alpha_2 \theta_2) \, d\theta_2 = \theta_1 F(\frac{b_1}{\alpha_2}) - \alpha_2 \frac{b_1}{\alpha_2} \int_0^{b_1/\alpha_2} f(\theta_2) \, d\theta_2$$

differentiating with respect to $b_1$, we get

$$\theta_1 \frac{1}{\alpha_2} f(\frac{b_1}{\alpha_2}) - \alpha_2 \frac{b_1}{\alpha_2} f(\frac{b_1}{\alpha_2}) \cdot \frac{1}{\alpha_2^2} = 0$$

$$\Rightarrow \frac{1}{\alpha_2} f(\frac{b_1}{\alpha_2}) (b_1 - \theta_1) = 0 \Rightarrow b_1 = \theta_1 \text{(similar for } b_2), \text{if } f(\frac{b_1}{\alpha_2}) > 0$$

For any independent, positive prior, bidding true type is a BE of the induced Bayesian game in second price auction.