# Mechanism Design (Inverse Game Theory)

The objectives / desired outcomes are set - task is to set the rules of the game

E.g., Election, license scarce resource (spectrum, cloud), matching students to universities

**General model:**

$N$ : set of players

$X$ : set of outcomes, e.g., winner in an election, which resource allocated to whom etc.

$\Theta_i$ : set of private information of agent $i$ (type). A type $\theta_i \in \Theta_i$

The type may manifest in the preferences over the outcomes in different ways

1. Ordinal: $\theta_i$ defines an ordering over the outcomes
2. Cardinal: an utility function $u_i$ maps an (outcome, type) pair to real numbers, $u_i : X \times \Theta_i \to \mathbb{R}$ (private value model)

   or $u_i : X \times \Theta \to \mathbb{R}$ (interdependent value model)

**Examples:** Voting: $X$ is the set of candidates

$\theta_i$ is a ranking over this candidates, e.g., $\theta_i = (a, b, c)$, i.e., $a$ is more preferred than $b$ which in turn is more preferred than $c$.

Single object allocation: an outcome is $x = (\underline{a}, \underline{p}) \in X$

$\underline{a} = (a_1, a_2, \cdots, a_n)$, $a_i \in \{0,1\}$, $\sum_{i \in N} a_i \leq 1$, allocations

$\underline{p} = (p_1, p_2, \dots, p_n)$, $p_i$ is the payment charged to $i$

$\theta_i$ : value of $i$ for the object

$$u_i(x, \theta_i) = a_i \theta_i - p_i .$$

But the designer has an objective

This is captured through a Social Choice Function (SCF)

$$f : \Theta_1 \times \Theta_2 \times \cdots \times \Theta_n \rightarrow X$$

E.g., in voting, if there is a candidate who beats everyone else in pairwise contests must be chosen as a winner.

in public project choice, where $\theta_i : X \rightarrow \mathbb{R}$, value for each project pick $f(\theta) \in \underset{a \in X}{\operatorname{argmax}} \sum_{i \in N} \theta_i(a)$.

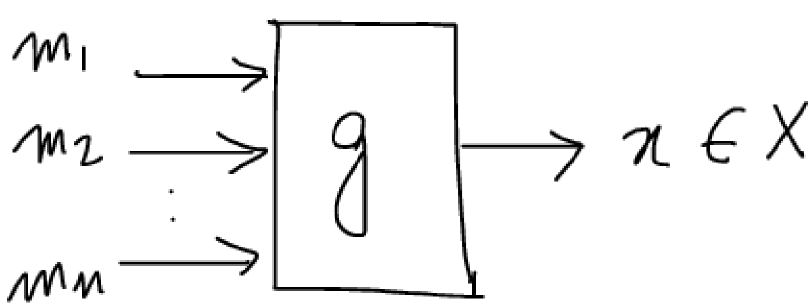
Q: How can we create a game where $f(\theta)$ emerges as an outcome of an equilibrium?

A: we need mechanisms.

<u>Defn</u>. An (indirect) mechanism is a collection of message spaces and a decision rule $\langle M_1, M_2, \ldots, M_n, g \rangle$

- $M_i$ is the message space of agent $i$
- $g : M_1 \times M_2 \times \cdots \times M_n \rightarrow X$

A direct mechanism is same as above with $M_i = \Theta_i$, $\forall i \in N$, $g \equiv f$.




The message space is similar to equipping every agent with a card deck and asking to pick some

Q: Why these are not so commonplace?

A: due to a result that will follow.

**Defn.** In a mechanism $\langle M_1, \ldots, M_n, g \rangle$, a message $m_i$ is weakly dominant for player $i$ at $\theta_i$ if

$$u_i(g(m_i, \tilde{m}_{-i}), \theta_i) \geqslant u_i(g(m_i', \tilde{m}_{-i}), \theta_i), \forall \tilde{m}_{-i}, \forall m_i'$$

[all subsequent definitions assume cardinal preferences, however they can be replaced with ordinal, e.g., the above one could be defined as

$$g(m_i, \tilde{m}_{-i}) \ \theta_i \ g(m_i', \tilde{m}_{-i}) \ \forall m_i', \forall \tilde{m}_{-i} \quad ]$$

this outcome is preferred at least as much as the latter

**Defn.** An SCF $f : \Theta \to X$ is implemented in dominant strategies by $\langle M_1, \ldots, M_n, g \rangle$ if

1. $\exists$ message mappings $s_i : \Theta_i \to M_i$, s.t., $s_i(\theta_i)$ is a dominant strategy for agent $i$ at $\theta_i$, $\forall \theta_i \in \Theta_i$, $\forall i \in N$, and
2. $g(s_1(\theta_1), \cdots, s_n(\theta_n)) = f(\theta)$, $\forall \theta \in \Theta$.

We call this an indirect implementation, i.e., SCF $f$ is dominant strategy implementable (DSI) by $\langle M_1, \ldots, M_n, g \rangle$.

**Defn.** A direct mechanism $\langle \Theta_1, \ldots, \Theta_n, f \rangle$ is dominant strategy incentive compatible (DSIC) if

$$u_i(f(\theta_i, \tilde{\theta}_{-i}), \theta_i) \geqslant u_i(f(\theta_i', \tilde{\theta}_{-i}), \theta_i), \ \forall \theta_i, \theta_i', \tilde{\theta}_{-i} \quad \forall i \in N.$$

To find if an SCF $f$ is dominant strategy implementable, we need to search over all possible indirect mechanisms $\langle M_1, \ldots, M_n, g \rangle$

But luckily, there is a result that reduces the search space.