Aggregating opinions (not worrying about truthful revelation)

Can we create social preference orders from individual preferences?

Arrow's social welfare function setup

Finite set of alternatives, $A = \{a_1, a_2, \dots, a_m\}$

Finite set of players, $N = \{1, \dots, n\}$

Each player i has a preference order R_i over A [a binary relation over A]. $a R_i b \Rightarrow a$ is at least as good as b.

Properties of R_i:

1. Completeness: for every pair of alternatives $a, b \in A$, either
   $a R_i b$ or $b R_i a$ or both
2. Reflexivity: $\forall a \in A$, $a R_i a$
3. Transitivity: if $a R_i b$ and $b R_i c$, then $a R_i c$, $\forall a, b, c \in A$ and $i \in N$.

Set of all preference ordering is $\mathcal{R}$

An ordering is linear if for every $a, b \in A$ s.t. $a R_i b$ and $b R_i a$, it holds that $a = b$. [indifferences are not allowed]

Set of all linear orderings is $\mathbb{P}$

Hence any arbitrary ordering R_i can be decomposed into (a) asymmetric part P_i, and (b) symmetric part I_i

E.g., $R_i = \begin{bmatrix} a \\ b, c \\ d \end{bmatrix} = \{(a,b), (a,c), (b,c), (c,b), (b,d), (c,d)\}$

$\Rightarrow P_i = \begin{bmatrix} a & a \\ b \;, & c \\ d & d \end{bmatrix} = \{(a,b), (a,c), (b,d), (c,d)\}$, $I_i = \{(b,c), (c,b)\}$

Arrovian Social Welfare Function (ASWF)

$F : \mathcal{R}^n \to \mathcal{R}$, domain and range both are rankings

motivation: The collective ordering of the society - if the most preferred is not feasible, the society can move to the next and so on.

$F(R) = F(R_1, R_2, \cdots, R_n)$ is an ordering

$\hat{F}(R)$ is the asymmetric part of $F(R)$

$\bar{F}(R)$ is the symmetric part of $F(R)$

Defn: Weak Pareto

An ASWF F satisfies weak Pareto if $\forall a, b \in A$

$$[a P_i b, \forall i \in N] \Rightarrow [a \hat{F}(R) b]$$

This notation is read as "whenever <the condition inside> holds, the implication follows"

$\forall R \in \mathcal{R}^n$, if $a P_i b, \forall i \in N$, then $a \hat{F}(R) b$.

There could be R's where the if condition doesn't hold, there the implication is vacuously true.

Defn. Strong Pareto

An ASWF F satisfies strong Pareto if $\forall a, b \in A$

$$[a R_i b, \forall i \in N, \text{and } a P_j b, \exists j] \Rightarrow [a \hat{F}(R) b]$$

Q: Which property implies the other?

We say $R_i, R_i' \in \mathcal{R}$ agree on $\{a, b\}$ if for agent i

$$a P_i b \Leftrightarrow a P_i' b, \quad b P_i a \Leftrightarrow b P_i' a, \quad a I_i b \Leftrightarrow a I_i' b$$

We use the shorthand $R_i|_{a,b} = R_i'|_{a,b}$ to denote this

If this holds for every agent, $R|_{a,b} = R'|_{a,b}$.

Defn. An ASWF F satisfies IIA if $\forall a, b \in A$,

$$[R|_{a,b} = R'|_{a,b}] \Rightarrow [F(R)|_{a,b} = F(R')|_{a,b}]$$

If the relative positions of two alternatives are same in two different preference profiles, then the aggregate must also retain the same relative positions.

Example:

R					R'			
a	a	c	d		d	c	b	b
b	c	b	c		a	a	c	a
c	b	a	b		b	b	a	d
d	d	d	a		c	d	d	c

Consider scoring rules $(s_1, s_2, \ldots, s_m)$, $s_i \geq s_{i+1}$, $i = 1, \ldots, m-1$

one special rule: plurality $s_1 = 1$, $s_2 = \cdots = s_m = 0$.

Does plurality satisfy IIA?

check $a\, F^{Pl}(R)\, b$, but $b\, F^{Pl}(R')\, a$, $R|_{a,b} = R'|_{a,b}$

Does dictatorship satisfy IIA?

Theorem (Arrow 1951)

Assume $|A| \geq 3$, if an ASWF F satisfies WP and IIA, then it must be dictatorial.