Arrow's impossibility result

Theorem (Arrow 1951)
Assume $|A| \geq 3$, if an ASWF F satisfies WP and IIA, then it must be dictatorial.

For the proof, we need the notions of decisiveness.

<u>Defn</u>. Let $F : \mathcal{R}^n \to \mathcal{R}$ be given, $G \subseteq N$, $G \neq \phi$

① G is almost decisive over $\{a, b\}$ if

$$[\, a P_i b, \forall i \in G, \text{ and } \; b P_j a \;\; \forall j \notin G \,] \Rightarrow [\, a \hat{F}(R)\, b \,]$$

We write this with the shorthand $\bar{D}_G(a,b)$: G is almost decisive over $\{a,b\}$ w.r.t. F

② G is decisive over $\{a,b\}$ if

$$[\, a P_i b \,, \forall i \in G \,] \Rightarrow [\, a \hat{F}(R)\, b \,]$$

Shorthand $D_G(a,b)$: G is decisive over $\{a,b\}$ w.r.t. F

Clearly, $D_G(a,b) \Rightarrow \bar{D}_G(a,b)$

The proof of the theorem proceeds in two parts

Part 1: Field expansion lemma
if a group is decisive over a pair of alternatives, it is decisive over all pairs of alternatives.

Part 2 : Group contraction lemma
if a group is decisive, then a strict subset of that group is also decisive.

Note that, these two lemmas immediately proves the theorem.

Part I: Field expansion lemma

Let F satisfy WP and IIA, then $\forall\, a,b,x,y$, $G \subseteq N, G \neq \phi, a\neq b, x\neq y$

$$\bar{D}_G(a,b) \Rightarrow D_G(x,y).$$

Remark: under WP and IIA, the two notions of decisiveness are identical.

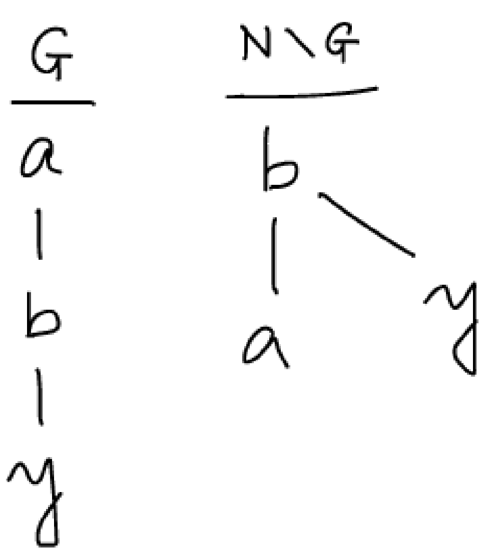
Proof: Cases to consider

1. $\bar{D}_G(a,b) \Rightarrow D_G(a,y)$, i.e., $x=a$, $y\neq a,b$
2. $\bar{D}_G(a,b) \Rightarrow D_G(x,b)$, i.e., $x\neq a,b$, $y=b$
3. $\bar{D}_G(a,b) \Rightarrow D_G(x,y)$, i.e., $x\neq a,b$, $y\neq a,b$
4. $\bar{D}_G(a,b) \Rightarrow D_G(x,a)$, i.e., $x\neq a,b$, $y=a$
5. $\bar{D}_G(a,b) \Rightarrow D_G(b,y)$, i.e., $x=b$, $y\neq a,b$
6. $\bar{D}_G(a,b) \Rightarrow D_G(a,b)$
7. $\bar{D}_G(a,b) \Rightarrow D_G(b,a)$

Case 1: $\bar{D}_G(a,b) \Rightarrow D_G(a,y)$, i.e., pick arbitrary $R \in \mathcal{R}^n$ s.t.

$a P_i y \;\forall i\in G$, need to show that $a\hat{F}(R)y$.

Construct R'



ensure $R_i'|_{a,y} = R_i|_{a,y}$, $\forall i \in N$

$\bar{D}_G(a,b) \Rightarrow a\hat{F}(R')b$

WP over $b,y \Rightarrow b\hat{F}(R')y$

transitivity $\Rightarrow a\hat{F}(R')y$

$\underset{IIA}{\Rightarrow} a\hat{F}(R)y$. Hence $D_G(a,y)$

Case 2: $\bar{D}_G(a,b) \Rightarrow D_G(x,b)$

Pick arbitrary R s.t. $x P_i b$, $\forall i \in G$. Need to show $x \hat{F}(R) b$.

$$R' \qquad \begin{array}{c|cc} \underline{G} & \underline{N \setminus G} & \\ x & x & b \\ a & & a \\ b & & \end{array}$$

Ensure $R'_i|_{x,b} = R_i|_{x,b}$ $\forall i \in N$.

$\bar{D}_G(a,b) \Rightarrow a \hat{F}(R') b$

WP on $x, a \Rightarrow x \hat{F}(R') a$

transitivity $\Rightarrow x \hat{F}(R') b \overset{IIA}{\Rightarrow} x \hat{F}(R) b$.

Case 3: $\bar{D}_G(a,b) \Rightarrow D_G(a,y)$ [case 1]

$\Rightarrow \bar{D}_G(a,y)$ [definition]

$\Rightarrow D_G(x,y)$ [case 2]

Case 4: $\bar{D}_G(a,b) \Rightarrow D_G(x,b)$ [case 2] $\quad x \neq a,b$

$\Rightarrow \bar{D}_G(x,b)$ [definition]

$\Rightarrow D_G(x,a)$ [case 1]

Case 5: $\bar{D}_G(a,b) \Rightarrow D_G(a,y)$ [case 1] $\quad y \neq a,b$

$\Rightarrow \bar{D}_G(a,y)$ [definition]

$\Rightarrow D_G(b,y)$ [case 2]

Case 6: $\bar{D}_G(a,b) \Rightarrow D_G(x,b)$ [case 2] $\quad x \neq a,b$

$\Rightarrow \bar{D}_G(x,b)$ [definition]

$\Rightarrow D_G(a,b)$ [case 2]

Case 7: $\bar{D}_G(a,b) \Rightarrow D_G(b,y)$ [case 5] $\quad y \neq a,b$

$\Rightarrow \bar{D}_G(b,y)$ [definition]

$\Rightarrow D_G(b,a)$ [case 1]

Part 2: Group contraction lemma

Let F satisfy WP and IIA. Let $G \subseteq N$, $G \neq \phi$, $|G| \geq 2$, be decisive. Then $\exists\, G' \subset G$, $G' \neq \phi$ which is also decisive.

Proof: If $|G| = 1$, nothing to prove. WLOG assume $|G| \geq 2$

Let $G_1 \subset G$, $G_2 = G \setminus G_1$, construct R

G_1	G_2	$N \setminus G$
a	c	b
b	a	c
c	b	a

$a P_i b \quad \forall i \in G$

and G decisive

$\Rightarrow \; a \hat{F}(R)\, b$ — (1)

Case 1: $a \hat{F}(R)\, c$, now consider G_1

$a P_i c \;\; \forall i \in G_1, \; c P_i a \;\; \forall i \notin G_1$

Consider all R', where this holds, by IIA $a \hat{F}(R')\, c$

hence $\bar{D}_G(a,c) \overset{FEL}{\Longrightarrow} G_1$ is decisive

Case 2: $\neg(a \hat{F}(R)\, c) \Rightarrow c\, F(R)\, a$

from (1) we get $a \hat{F}(R)\, b \underset{trans}{\Rightarrow} c \hat{F}(R)\, b$

Consider G_2,

$c P_i b \;\; \forall i \in G_2$, and $b P_i c \;\; \forall i \notin G_2$

using IIA as before $\bar{D}_{G_2}(b,c) \overset{FEL}{\Longrightarrow} G_2$ is decisive

This concludes the proof.