Arrovian social welfare setup is too demanding

It says that achieving a "social ordering" in a democratic way is impossible

Steps to mitigate:

① Consider a social choice setting – instead of an ordering, select an alternative.

② Put restrictions on agents' preferences

Social Choice Function

$$f : \mathbb{P}^n \to A \text{, assuming only strict preferences}$$

most representative example: voting

Various voting rules

① Scoring rule: $(s_1, s_2, \ldots, s_m)$ common score vector. Every voter's k^{th} preferred alternative is given a score of s_k. Scores are summed for each candidate – highest score wins.

- special cases: plurality – $(1, 0, \cdots, 0)$
  - veto – $(1, 1, \cdots, 1, 0)$
  - Borda – $(m-1, m-2, \ldots, 0)$
  - harmonic – $(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{m})$
  - k-approval – $(\underbrace{1, \cdots, 1}_{k}, 0, \cdots, 0)$

② Plurality with runoff: two phases – first, top 2 highest scored candidates remain and the voters vote again [French presidential]

③ Maximin: candidate with largest margin of victory wins

④ Copeland: based on score = # of wins in pairwise elections

A Condorcet winner is a candidate that beats every other candidate in pairwise election. It is not guaranteed to exist.

a	b	c
b	c	a
c	a	b

If a Condorcet winner exists, the voting rules that returns it as the winner are called "Condorcet consistent"

Easy to check that Copeland is Condorcet consistent (by design)

But plurality is not

30%	30%	40%
a	b	c
b	a	a
c	c	b

pairwise election
a beats b 70-30
a beats c 60-40
a is the Condorcet winner

Actually, no scoring rule is Condorcet consistent.

Back to Social Choice Functions: $f: \mathbb{P}^n \to A$

Pareto domination: An alternative a is Pareto dominated by b if $\forall i \in N$ $b P_i a$. [it is Pareto dominated if some such b exists]

Pareto Efficiency: An SCF f is PE if for every preference profile P and $a \in A$, if a is Pareto dominated, then $f(P) \neq a$.

Unanimity: An SCF f is UN if for every preference profile P having $P_1(1) = P_2(1) = \cdots = P_n(1) = a$ [where $P_i(k)$ is the k^{th} preferred alternative of i], $f(P) = a$.

Clearly $PE \subset UN$, when the top candidate is the same a for all agents, all other alternatives are Pareto dominated by a. Hence a PE SCF can choose nothing but a.

Why strict? consider a profile where the top alternative is not the same, a UN SCF can pick a dominated alternative

Ontoness: An SCF is ONTO if $\forall a \in A$, $\exists P^{(a)} \in \mathcal{P}^n$ s.t. $f(P^{(a)}) = a$.

Claim: $UN \subset ONTO$

Manipulability: An SCF f is manipulable if $\exists\, i \in N$ and a profile P s.t. $f(P_i', P_{-i})\, P_i\, f(P_i, P_{-i})$ for some P_i'.

Ex. Plurality (tie breaking in favor of $a > b > c$)

a	b	c
b	a	b
c	c	a
4	4	1

← true votes

last voter should vote for b

Copeland (tie breaking in favor of $a > b > c$)

a	b	c
b	c	a
c	a	b
1	1	1

each candidate has Copeland score $= 1$, a is the winner.

a is least preferred

if reports c, b, a then c is the Copeland winner.

An SCF is strategyproof if it is not manipulable by any agent at any profile.

Implications of strategyproofness

Defn: Dominated set of a at preference P_i

$$D(a, P_i) = \{ b \in A : a\, P_i\, b \}$$

The set of alternatives below a in that preference

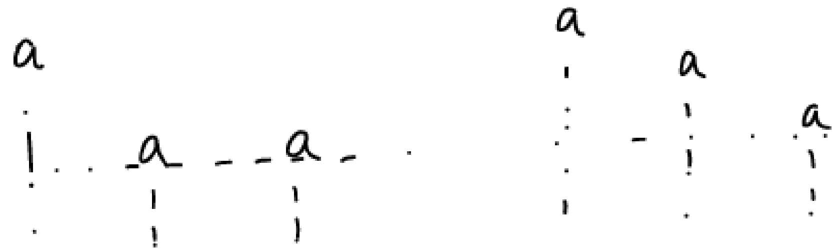
e.g., $P_i = \begin{matrix} b \\ d \\ a \\ c \end{matrix} \Rightarrow D(d, P_i) = \{a, c\}$

Monotonicity: An SCF f is monotone if for any two profiles P and P' with $f(P) = a$ and

$$D(a, P_i) \subseteq D(a, P_i'), \forall i \in N, \text{ must imply}$$

$$f(P') = a.$$

The relative position of a has weakly improved from R to R'. This property says if a was the outcome in P, then it must continue to be the outcome in P'



P if $f(P) = a$, then P'

$f(P') = a$

Theorem: An SCF f is strategyproof (SP) iff it is monotone (MONO).