Corollary: f is SP+PE $\Leftrightarrow$ f is SP+UN $\Leftrightarrow$ f is SP+ONTO

Gibbard-Satterthwaite Theorem (G73, S75)

Suppose $|A| \geq 3$, f is ONTO and SP iff f is dictatorial.

The statements with f is PE/UN and SP are equivalent.

Few points to note:

① $|A| = 2$: GS theorem does not hold. Plurality with a fixed tie breaking rule is SP, ONTO, and non-dictatorial.

② The domain is $\mathcal{P}$: all permutations of the alternatives are feasible. Intuitively, every voter has many options to misreport. If the domain was limited, then GS may not hold.

③ Indifference in preferences : in general, GS theorem does not hold. In the proof, we use some specific constructions. If they are possible, then GS theorem holds.

④ Cardinalization : GS theorem will hold as long as all possible ordinal ranks are feasible in the cardinal preferences.

For the proof, we will follow a direct approach (Sen 2001). First prove for $n=2$ and then apply induction on the number of agents.

Lemma: Suppose $|A| \geq 3$, $N = \{1,2\}$, f is ONTO and SP, then for every preference profile P, $f(P) \in \{P_1(1), P_2(1)\}$.

Proof: If $P_1(1) = P_2(1)$, then unanimity implies $f(P) = P_1(1)$ (Corollary above)

Say $P_1(1) = a \neq b = P_2(1)$. For contradiction assume $f(P) = c \neq a, b$ (need 3 alt)

P_1	P_2	P_1	P_2'	P_1'	P_2'	P_1'	P_2
a	b	a	b	a	b	a	b
⋮	⋮	⋮	a	b	a	b	⋮
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

outcome here is c

Now $f(P_1 P_2') \in \{a, b\}$ [because all other alternatives except b is Pareto dominated by a]

But if $f(P_1 P_2') = b$, then player 2 manipulates from P_2 to P_2'. Hence, $f(P_1 P_2') = a$.

By a similar argument, $f(P_1' P_2) = b$

But now MONO will lead to a contradiction

$P_1' P_2 \to P_1' P_2'$, outcome should be b

$P_1 P_2' \to P_1' P_2'$, outcome should be a $\square$

Lemma: Suppose $|A| \geq 3$, $N = \{1,2\}$, f is ONTO and SP.

Let P: $P_1(1) = a \neq b = P_2(1)$, P': $P_1'(1) = c$, $P_2'(1) = d$.

If $f(P) = a$, then $f(P') = c$

If $f(P) = b$, then $f(P') = d$.

This proves dictatorship for two players.

Proof: If $c = d$, unanimity proved the lemma. Hence consider $c \neq d$.

Cases	c	d
1	a	b
2	$\neq a, b$	b
3	$\neq a, b$	$\neq b$
4	a	$\neq a, b$
5	b	$\neq a, b$
6	b	a

These cases are exhaustive

Enough to consider the case

If $f(P) = a \Rightarrow f(P') = c$

The other case is symmetric

Case 1: $c = a$, $d = b$,

P_1	P_2	P_1'	P_2'	$\hat{P}_1$	$\hat{P}_2$
a	b	a	b	a	b
$\vdots$	$\vdots$	$\vdots$	$\vdots$	b	a
				$\vdots$	$\vdots$

We know (by previous lemma) $f(P') \in \{a, b\}$

say for contradiction $f(P') = b$

$P_1 P_2 \xrightarrow[\text{MONO}]{} \hat{P}_1 \hat{P}_2$

$a \qquad\qquad a$

$P_1' P_2' \xrightarrow[\text{MONO}]{} \hat{P}_1 \hat{P}_2$

$b \qquad\qquad b$

Case 2: $c \neq a, b,\ d = b$

P_1	P_2	P_1'	P_2'	$\hat{P}_1$	P_2
a	b	c	b	c	b
⋮	⋮	⋮	⋮	a	⋮
				⋮	

$f(P') \in \{c, b\}$

assume $f(P') = b$ (for contradiction)

$P_1'\ P_2' \rightarrow \hat{P}_1\ P_2$ (apply case 1)

$b \qquad\qquad b$

agent 1 misreports $\hat{P}_1 \rightarrow P_1$ as $a\ \hat{P}_1\ b$.

Case 3: $c \neq a, b,\ d \neq b$

P_1	P_2	P_1'	P_2'	$\hat{P}_1$	$\hat{P}_2$
a	b	c	d	c	b
⋮	⋮	⋮	⋮	⋮	⋮

Say $f(P') = d$

$P' \rightarrow \hat{P} \quad f(\hat{P}) = b$ (case 2)

$P \rightarrow \hat{P} \quad f(\hat{P}) = c$ (case 2)

Case 4: $c = a,\ d \neq b, a$

P_1	P_2	P_1'	P_2'	$\hat{P}_1$	$\hat{P}_2$
a	b	c=a	d	a	b
⋮	⋮	⋮	⋮	⋮	⋮

Say $f(P') = d$

$P' \rightarrow \hat{P} \quad f(\hat{P}) = b$ (case 2)

$P \rightarrow \hat{P} \quad f(\hat{P}) = a$ (case 1)

Case 5: $c = b,\ d \neq b, a$

P_1	P_2	P_1'	P_2'	$\hat{P}_1$	$\hat{P}_2$
a	b	c=b	d	c	d
⋮	⋮	⋮	⋮	⋮	⋮

Say $f(P') = d$

$P' \rightarrow \hat{P} \quad f(\hat{P}) = d$ (case 4)

$P \rightarrow \hat{P} \quad f(\hat{P}) = a$ (case 4)

Case 6: $c=b$, $d=a$

$f(P') = a$

$x \neq a, b$

P_1	P_2	P_1'	P_2'	$\hat{P}_1$	P_2'	$\tilde{P}_1$	P_2'
a	b	$c=b$	$d=a$	b	a	x	a
$\vdots$	$\vdots$	$\vdots$	$\vdots$	x	$\vdots$		
				$\vdots$			

$P' \to (\hat{P}_1\ P_2')$, $f(\hat{P}_1\ P_2') = a$ (case 1)

$P \to (\tilde{P}_1\ P_2')$, $f(\tilde{P}_1\ P_2') = x$ (case 3)

Player 1 manipulates from $\hat{P}_1\ P_1' \to \tilde{P}_1\ P_2'$

since $x\ \hat{P}_1\ a$

More than 2 agents $\to$ induction on the number of agents. See Sen (2001): "A direct proof of GS theorem".