Equivalence of SP, ONTO, ANON and median voting rule in single peaked domain

Theorem (Moulin 1980)

A strategyproof SCF is onto and anonymous iff it is a median voter SCF.

Proof: $\Leftarrow$ median voter SCF is SP (previous theorem).

It is anonymous, if we permute the agents with peaks unchanged.

The outcome does not change.

It is onto, pick any arbitrary alternative $a$, put peaks of all players at $a$. The outcome will be a irrespective of the positions of the phantom peaks - since there are $(n-1)$ phantom peaks and $n$ agent peaks.

$\Rightarrow$ Given, $f: A^n \rightarrow A$ is SP, ANON, and ONTO.

Define, $P_i^0$: agent $i$'s preference with peak at leftmost $\Rightarrow$ $\Rightarrow$

$P_i^1$: agent $i$'s preference with peak at rightmost $\Rightarrow$

The proof is constructive, we will construct the median voting rule (which needs the phantom peaks s.t. the outcome of an arbitrary $f$ matches the outcome of the median SCF.
First, construct phantom peaks

\[ y_j = f\left( P_1^0, P_2^0, \ldots, P_{n-j}^0, P_{n-j+1}^{-1}, \ldots, P_n^{-1}\right), \ j=1, \ldots, n-1 \]

\( (n-j) \text{ peaks left-most} \quad j \text{ peaks right-most} \)

which agents have which peaks does not matter because of anonymity.

Claim: \( y_j \leq y_{j+1}, \ j=1, \ldots, n-2, \) i.e., peaks are non-decreasing.

Proof: \( y_{j+1} = f\left( P_1^0, P_2^0, \ldots, P_{n-j}^1, P_{n-j+1}^{-1}, \ldots, P_n^{-1}\right) \)

Due to SP, \( y_j P_{n-j} P_{n-j+1} \) on. They are same with peak at 0, hence \( y_j \leq y_{j+1}. \) \( \square \)

Consider an arbitrary profile, \( P = (P_1, P_2, \ldots, P_n), \)
\( P_i(1) = p_i \) (the peaks).

Claim: Suppose \( f \) satisfies SP, ONTO, ANON, then

\[ f(P) = \text{median}(p_1, \ldots, p_n, y_1, \ldots, y_{n-1}) \]

WLOG, can assume \( p_1 \leq p_2 \leq \ldots \leq p_n \) due to ANON.
also say, \( a = \text{median}(p_1, \ldots, p_n, y_1, \ldots, y_{n-1}) \)
Case 1: $a$ is a phantom peak
Say $a = y_j$, for some $j \in \{1, 2, \ldots, n-1\}$.
This is a median of $2n-1$ points, of which $(j-1)$ phantom peaks lie on the left (see the claim before). Rest $(n-j)$ points are agent peaks.

\[
\begin{array}{c|c}
(j-1) \text{phantom} & (n-1-j) \text{phantom} \\
(n-j) \text{agent} & y_j & j \text{ agent}
\end{array}
\]

Hence, $p_1 \leq \cdots \leq p_{n-j} \leq y_j = a \leq p_{n-j+1} \leq \cdots \leq p_n$.

Use a similar transformation as we used earlier.

\[
f(P_1^0, P_2^0, \ldots, P_{n-j}^0, P_{n-j+1}^1, \ldots, P_n^1) = y_j \quad \text{(definition)}
\]

\[
f(P_1, P_2^0, \ldots, P_{n-j}^0, P_{n-j+1}^1, \ldots, P_n^1) = b \quad \text{(say)}
\]

By SP, $y_j \leq b \Rightarrow y_j \leq b$

again by SP, $b \leq y_j$, but $p_i \leq y_j \Rightarrow b \leq y_j$
hence $b = y_j$

repeat this argument for first $(n-j)$ agents to get
\[ f(P_1, P_2, \ldots, P_{n-j}, P_{n-j+1}, \ldots, P_n) = y_j \]

Now consider

\[ f(P_1, P_2, \ldots, P_{n-j}, P_{n-j+1}, \ldots, P_n) = b \text{ (say)} \]

Apply very similar argument

\[ y_j P_n b \Rightarrow b \leq y_j \]

\[ b \leq y_j \text{ and } y_j \leq P_n \Rightarrow y_j \leq b \]

Hence

\[ f(P_1, \ldots, P_n) = y_j \]