Claim: Suppose $f$ satisfies SP, ONTO, ANON, then
\[ f(P) = \text{median}(p_1, \ldots, p_n, y_1, \ldots, y_{n-1}). \]

Case 1: $a$ is a phantom peak - proved

Case 2: $a$ is an agent peak

We will prove this for 2 players. The general case repeats this argument.

Claim: $N = \{1, 2\}$, let $P$ and $P'$ be such that
\[ p_i(1) = p_i'(1), \forall i \in N. \] Then $f(P) = f(P')$.

Proof: Let $a = p_1(1) = p_1'(1)$, and $p_2(1) = p_2'(1) = b$.
\[ f(P) = x \quad \text{and} \quad f(P, P_2') = y. \]

Since $f$ is SP, $x \, P_1 \, y$ and $y \, P_1' \, x$

Since peaks of $P_1$ and $P_1'$ are the same, if $x, y$ are on the same side of the peak, they must be the same, as the domain is single peaked.

The only other possibility is that $x$ and $y$ fall on different sides of the peak. We show that this is impossible.

WLOG $x < a < y$ and $a < b$. 
\( f \circ SP \leftrightarrow f \circ SP \oplus PE \)

PE requires \( f(P) \in [a, b] \), but \( f(P) = \alpha < a \) \( \Rightarrow \)

now repeat this argument for \( (P_1', P_2) \rightarrow (P_1', P_2') \) \( \square \)

Profile: \( (P_1, P_2) = P \), \( P_1(1) = a \), \( P_2(1) = b \)

\( y_1 \) is the phantom peak.

by assumption, median \((a, b, y_1)\) is an agent peak

WLOG assume the median is \( a \).

Assume for contradiction \( f(P) = c \neq a \).

By PE, \( c \) must be within \( a \) and \( b \). We have two cases to consider: \( b < a < y_1 \) and \( y_1 < a < b \).

Case 2.1: \( b < a < y_1 \), by PE \( c < a \)

construct \( P_1' \), s.t.: \( P_1'(1) = a = P_1(1) \)

and \( y_1, P_1'c \) (possible since they are on different sides of \( a \)).

by the earlier claim, \( f(P) = c \Rightarrow f(P_1', P_2) = c \).

now consider the profile \((P_1', P_2)\)

\( \uparrow \) peak at the rightmost
\( p_2(1) = b < y_1 \leq p_1'(1) \), hence the median of \( \{ b, y_1, p_1'(1) \} \)

is \( y_1 \) (which is a phantom peak, hence case 1 applies).

\[ f(p_1', p_2) = y_1. \]

But \( y_1, p_1' \) (by construction) and \( f(p_1', p_2) = c \)

agent 1 manipulates \( p_1' \rightarrow p_1' \), contradiction to \( f \) being SP.

**Case 2.2:** \( y_1 < a < b, \ PE \Rightarrow a < c \)

construct \( p_1' \) s.t. \( p_1'(1) = a = p_1(1) \) and \( y_1, p_1' \)

\[ f(p_1', p_2) = c \quad (\text{by claim}) \]

consider \( (p_1^0, p_2) \), \( p_1^0(1) \leq y_1 < b \Rightarrow f(p_1^0, p_2) = y_1 \),

but \( y_1, p_1' \), hence manipulable by agent 1.

This completes the proof for two agents (Case 2). For the generalization to \( n \) players, see Moulin (1980)

"On strategyproofness and single peakedness".