Task allocation domain (related but different than single-peaked)

Unit amount of task to be shared among $n$ agents

Agent $i$ gets a share $x_i \in [0,1]$ of the job, $\sum x_i = 1$.

Agent payoff: every agent has a most preferred share of work.

Example: The task has rewards - wages per unit time = $w$

if agent $i$ works for $t_i$ time then gets $w t_i$

the task also has costs, e.g., physical tiredness / less free time etc.

let the cost is quadratic = $c_i t_i^2$

net payoff = $w t_i - c_i t_i^2$ $\Rightarrow$ maximized at $t_i^* = \frac{W}{2c_i}$

and monotone decreasing on both sides.

This is single-peaked over the share of the task and $w t_i$ over the alternatives. Suppose, two alternatives are $(0.2, 0.4, 0.4)$ and $(0.2, 0.6, 0.2)$ - player 1 likes both of them equally.

There can't be a single common order over the alternatives s.t.
The preferences are single-peaked for all.

Denote this domain of task allocation with $T$ (single peaked over task share)

SCF: $f: T^n \to A$

Let $p \in T^n$, $f(p) = (f_1(p), f_2(p), \ldots, f_n(p))$

$f_i(p) \in [0,1], \forall i \in N; \sum_{i \in N} f_i(p) = 1$

Player $i$ has a peak $p_i$ over the share of task.

**Pareto Efficiency:** An SCF $f$ is PE if there does not exist another share of task that is weakly preferred by all agents and strictly preferred by at least one, i.e.,

$$\forall a \in A \text{ s.t. } a_R i f(P), \forall i \in N \text{ and } \exists j \text{ s.t. } a_P j f(P)$$

**Implications:**

1. $\sum_{i \in N} p_i = 1$, allocate tasks according to the peaks of the agents. This is the unique PE.

2. $\sum_{i \in N} p_i > 1$, $\exists K \in N \text{ s.t. } f_K(P) < p_K$.

Q: Can there be an agent $j$ s.t. $f_j(P) > p_j$ if $f$ is PE?

If so, increasing $K$'s share of task and reducing $j$'s makes both players strictly better off. Therefore

$$\forall j \in N, f_j(P) \leq p_j.$$

3. If $\sum_{i \in N} p_i < 1$, similarly $\forall j \in N, f_j(P) \geq p_j$.

**Anonymity:** (if agent preferences are permuted, the shares will also get permuted accordingly.

$$f_{\sigma(j)}(P^\sigma) = f_j(P)$$
\[ N = \{1,2,3\}, \quad \sigma(1) = 2, \quad \sigma(2) = 3, \quad \sigma(3) = 1 \]
\[ P = (0.7, 0.4, 0.3) \Rightarrow P^\sigma = (0.3, 0.7, 0.4) \]
\[ f_1 (0.7, 0.4, 0.3) = f_2 (0.3, 0.7, 0.4) \]

**Candidate SCFs:**

**Serial dictatorship:** A predetermined sequence of the agents is fixed. Each agent is given either his peak share or a leftover share. If \( \Sigma p_i < 1 \), then the last agent is given the leftover share.

**Properties:** PE, SP, but not ANON. Also quite unfair for the last agent.

**Proportional:** Every player is assigned a share that is \( c \) times their peak, s.t. \( c \Sigma p_i = 1 \)

overload if \( \Sigma p_i < 1 \), underload if \( \Sigma p_i > 1 \).

Q: Is it ANON, PE, SP?

Suppose peaks are 0.2, 0.3, 0.1 for 3 players, \( c = \frac{1}{6} \)
player 1 gets \( \frac{1}{3} \) (more than 0.2)
if the report is 0.1, 0.3, 0.1, \( c = \frac{1}{0.5} \), player 1 gets 0.2.