How to ensure PE, ANON, and SP in task allocation domain?

Uniform rule (Sprumont 1991)

Suppose, $\sum_{i \in N} p_i < 1$

Begin with everyone's allocation being 1. Keep reducing until $\sum f_i = 1$

Whenever some agent's peak is reached, set the allocation for that agent to be its peak.

Definition:

1. $f_i^u (P) = p_i$ if $\sum_{i \in N} p_i = 1$
2. $f_i^u (P) = \max \{ p_i, \mu (P) \}$ if $\sum_{i \in N} p_i < 1$

where $\mu (P)$ solves $\sum_{i \in N} \max \{ p_i, \mu^2 \} = 1$.

3. $f_i^u (P) = \min \{ p_i, \lambda (P) \}$ if $\sum_{i \in N} p_i > 1$

where $\lambda (P)$ solves $\sum_{i \in N} \min \{ p_i, \lambda^2 \} = 1$.

Q: Is this ANON, PE, and SP?

Theorem (Sprumont 1991)
The uniform rule SCF is ANON, PE, and SP.
Proof: ANON is obvious — only the peaks matter and not their owners.

PE: the allocation is s.t.

\[ f_i^u(p) = p_i, \forall i \in N, \text{ if } \sum p_i = 1 \]
\[ f_i^u(p) > p_i, \forall i \in N, \text{ if } \sum p_i < 1 \]
\[ f_i^u(p) \leq p_i, \forall i \in N, \text{ if } \sum p_i > 1 \]

for some players the peaks are allocated, and for others the allocation is the same. This is PE, since any other allocation can only improve the allocation of a player at the cost of another player’s allocation.

Strategy-proofness:

for case \( \sum p_i = 1 \), every agent gets their peak — no reason to deviate.

Case \( \sum p_i < 1 \), then \( f_i^u(p) > p_i, \forall i \in N \).

Only possible manipulation for agents that have \( f_i^u(p) > p_i \):

\[ \Rightarrow \mu(p) > p_i, \text{ i.e., the allocation stopped before reaching } p_i \]. The only way i can change the allocation is by reporting \( p_i' > \mu(p) > p_i \) — but this is a worse allocation for i than \( \mu(p) \).

Similar argument for case \( \sum p_i > 1 \). This completes the proof.
The converse is also true. We skip the proof.

**Theorem:** An SCF is SP, PE, and ANON iff it is the uniform rule.