Quasi linear preferences

The SCF is decomposed into two components

Allocation rule component

$$f : \Theta_1 \times \Theta_2 \times \cdots \times \Theta_n \to A$$

When the types are θ_i, $i \in N$, $f(\theta_1, \dots, \theta_n) = a \in A$

Payment function

$$p_i : \Theta_1 \times \Theta_2 \times \cdots \times \Theta_n \to \mathbb{R}, \quad \forall i \in N$$

When the types are θ_i, $i \in N$, $p_i(\theta_1, \dots, \theta_n) = \pi_i \in \mathbb{R}$

Examples of allocation rules

1. Constant rule, $f^c(\theta) = a \quad \forall \theta \in \Theta$

2. Dictatorial rule, $f^D(\theta) \in \underset{a \in A}{\arg\max}\ v_d(a, \theta_d)$ for some $d \in N$ $\forall \theta \in \Theta$.

3. Allocatively efficient rule / utilitarian rule

$$f^{AE}(\theta) \in \underset{a \in A}{\arg\max} \sum_{i \in N} v_i(a, \theta_i)$$

Note: this is different from Pareto efficiency (PE is a property defined for the outcome which also considers the payment)

4. Affine maximizer rule:

$$f^{AM}(\theta) \in \underset{a \in A}{\arg\max} \left(\sum_{i \in N} \lambda_i v_i(a, \theta_i) + K(a) \right), \quad \lambda_i \geq 0, \text{ not all zero.}$$

⑤ Max-min / egalitarian

$$f^{MM}(\theta) \in \underset{a \in A}{\arg\max} \min_{i \in N} v_i(a, \theta_i)$$

Examples of payment rules

① No deficit : $\sum_{i \in N} p_i(\theta) \geqslant 0 \quad \forall \theta \in \Theta$.

② No subsidy : $p_i(\theta) \geqslant 0$, $\forall \theta \in \Theta$, $\forall i \in N$.

③ Budget balanced : $\sum_{i \in N} p_i(\theta) = 0$, $\forall \theta \in \Theta$.

Recall : Incentive Compatibility

A mechanism is the tuple of the allocation and payment rule (f, p)

A mechanism (f, p) is dominant strategy incentive compatible (DSIC) if $\forall i \in N$

$$v_i(f(\theta_i, \tilde{\theta}_{-i}), \theta_i) - p_i(\theta_i, \tilde{\theta}_{-i}) \geqslant v_i(f(\theta_i', \tilde{\theta}_{-i}), \theta_i) - p_i(\theta_i', \tilde{\theta}_{-i}),$$

$$\forall \theta_{-i} \in \Theta_{-i}, \ \forall \theta_i, \theta_i' \in \Theta_i.$$

DSIC means truthtelling is a weakly DSE.

We say that the payment rule p implements f in dominant strategies OR f is implementable in dominant strategies (by a payment rule)

In QL domain, we are often more interested in the allocation rule than the whole SCF (includes payment).

What needs to be satisfied for a DSIC mechanism $(f, \underline{p})$?

$N = \{1,2\}$, $\Theta_1 = \Theta_2 = \{\theta^H, \theta^L\}$, $f : \Theta_1 \times \Theta_2 \to A$

the following conditions must hold

$$v_1(f(\theta^H, \theta_2), \theta^H) - p_1(\theta^H, \theta_2) \geqslant v_1(f(\theta^L, \theta_2), \theta^H) - p_1(\theta^L, \theta_2), \forall \theta_2$$

$$v_1(f(\theta^L, \theta_2), \theta^L) - p_1(\theta^L, \theta_2) \geqslant v_1(f(\theta^H, \theta_2), \theta^L) - p_1(\theta^H, \theta_2), \forall \theta_2$$

for player 2 :

$$v_2(f(\theta_1, \theta^H), \theta^H) - p_2(\theta_1, \theta^H) \geqslant v_2(f(\theta_1, \theta^L), \theta^H) - p_2(\theta_1, \theta^L), \forall \theta_1$$

$$v_2(f(\theta_1, \theta^L), \theta^L) - p_2(\theta_1, \theta^L) \geqslant v_2(f(\theta_1, \theta^H), \theta^L) - p_2(\theta_1, \theta^H), \forall \theta_1$$

Properties of the payment that implements an allocation rule

① Say $(f, \underline{p})$ is incentive compatible. Consider another payment

$$q_i(\theta_i, \underline{\theta}_{-i}) = p_i(\theta_i, \underline{\theta}_{-i}) + h_i(\underline{\theta}_{-i}) \quad \forall \theta, \forall i \in N.$$

Q: is $(f, \underline{q})$ DSIC ?

A: Yes.

$$v_i(f(\theta_i, \tilde{\underline{\theta}}_{-i}), \theta_i) - p_i(\theta_i, \tilde{\underline{\theta}}_{-i}) - h_i(\tilde{\underline{\theta}}_{-i})$$
$$\geqslant v_i(f(\theta_i', \tilde{\underline{\theta}}_{-i}), \theta_i) - p_i(\theta_i', \tilde{\underline{\theta}}_{-i}) - h_i(\tilde{\underline{\theta}}_{-i})$$
$$\forall \theta_i, \theta_i', \tilde{\underline{\theta}}_{-i}, \forall i \in N.$$

if we can find a payment that implements an allocation rule, there exists uncountably many payments that can implement it.

The converse question: when do the payments that implement f differ only by a factor $h_i(\underline{\theta}_{-i})$?

② Implication of incentive compatibility on payment

suppose the allocation is same in two type profiles θ and $\tilde{\theta} = (\tilde{\theta}_i, \underline{\theta}_{-i})$

$f(\theta) = f(\tilde{\theta}) = a$, then if $\underline{p}$ implements f, then

$p_i(\theta) = p_i(\tilde{\theta})$. [exercise]