Mechanism design for selling a single indivisible object

Motivation: simplest yet elegant results

Setup: type set of agent $i$: $T_i \subseteq \mathbb{R}$

$t_i \in T_i$ denotes the value of agent $i$ if she wins the object.

An allocation $a$ is a vector of length $n$ that represents the probability of winning the object by the respective agent. Hence,

Set of allocations: $\Delta A = \{ a \in [0,1]^n : \sum_{i=1}^n a_i = 1 \}$

Allocation rule: $f: T_1 \times T_2 \times \ldots \times T_n \rightarrow \Delta A$

Valuation: $v_i(a, t_i) = a_i \cdot t_i$ (product form) - expected valuation.

Hence, $f_i(t_i, t_{-i})$ is the probability of winning the object for agent $i$ when the type profile is $(t_i, t_{-i})$.

Recall: Vickrey/Second-price auction: type is $v_i$.

Define $t_i^{(2)} = \max_{j \neq i} \{ v_j \}$

Agent $i$ wins if $v_i > t_i^{(2)}$, loses if $v_i < t_i^{(2)}$.

A tie-breaking rule decides if equality.

Since, payment is $t_i^{(2)}$ if $i$ is the winner. The utility is zero in case of a tie.

$$u_i = \begin{cases} 0 & \text{if } v_i \leq t_i^{(2)} \\ v_i - t_i^{(2)} & \text{if } v_i > t_i^{(2)} \end{cases}$$
Observations:
1. Utility is convex, derivative is zero if $v_i < t_i^{(2)}$ and $1$ if $v_i > t_i^{(2)}$ — not differentiable at $v_i = t_i^{(2)}$.
2. Whenever differentiable, it coincides with the allocation probability.

Known facts from convex analysis (see, e.g., Rockafeller (1980))

Fact 1: Convex functions are continuous in the interior of their domain. Jumps can occur only at the boundaries.

Fact 2: Convex functions are differentiable "almost everywhere." The points where the function is not differentiable form a countable set (see the example before) — has measure zero.

Recall: A function $g : I \to \mathbb{R}$ (where $I$ is an interval) is convex if for every $x, y \in I$ and $\lambda \in [0, 1]$,

$$\lambda g(x) + (1-\lambda) g(y) \geq g(\lambda x + (1-\lambda)y).$$

If $g$ is differentiable at $x \in I$, we denote the derivative by $g'(x)$. The following definition extends the idea of gradient
Define: For any $x \in I$, $x^*$ is a subgradient of $g$ at $x$ if
\[ g(z) \geq g(x) + x^*(z-x) \quad \forall z \in I. \]

Few standard results (proofs: any standard text on convex analysis)

Lemma 1: Let $g: I \rightarrow \mathbb{R}$ be a convex function. Suppose $x$ is in the interior of $I$ and $g$ is differentiable at $x$. Then $g'(x)$ is the unique subgradient of $g$.

Lemma 2: Let $g: I \rightarrow \mathbb{R}$ be a convex function. Then for every $x \in I$ a subgradient of $g$ at $x$ exists.

Fact 3: Let $I' \subseteq I$ be the set of points where $g$ is differentiable. The set $I \setminus I'$ is of measure zero. The set of subgradients at a point forms a convex set.

Define $g'_+(x) = \lim_{z \rightarrow x, z \in I', z > x} g'(z)$, $g'_-(x) = \lim_{z \rightarrow x, z \in I', z < x} g'(z)$

Fact 4: The set of subgradients at $x \in I \setminus I'$ is $[g'_-(x), g'_+(x)]$. 
We will denote the set of subgradients of $g$ at $x \in I$ as $\partial g(x)$.

Lemma 1 says $\partial g(x) = \{ g'(x) \} \forall x \in I$.

Lemma 2 says that $\partial g(x) \neq \emptyset \forall x \in I$.

Lemma 3: Let $g : I \rightarrow \mathbb{R}$ be a convex function. Let $\phi : I \rightarrow \mathbb{R}$ be a subgradient function, i.e., $\phi(z) \in \partial g(z) \forall z \in I$.

Then for all $x, y \in I$ s.t. $x > y$, we have $\phi(x) \geq \phi(y)$.

$\phi(z)$ picks one value at every $z$ (even if subgradients can be many).

This result says that subgradient functions are monotone non-decreasing.

Lemma 4: Let $g : I \rightarrow \mathbb{R}$ be a convex function. Then for any $x, y \in I$,

$$g(x) = g(y) + \int_y^x \phi(z) \, dz,$$

where $\phi : I \rightarrow \mathbb{R}$ is s.t. $\phi(z) \in \partial g(z), \forall z \in I$. 