Monotonicity and Myerson's lemma

**Defn**: An allocation rule is non-decreasing if for every agent $i \in N$ and $t_i, t'_i \in T_i$ we have $f_i(t_i, t_i) > f_i(s_i, t'_i)$, $\forall s_i, t_i \in T_i$, $t_i > t'_i$.

Holding the types of other agents fixed, the probability of allocation never decreases with valuation.

**Theorem** (Myerson 1981)

Suppose $T_i = [0, b_i]$ $\forall i \in N$, and the valuations are in the product form.

An allocation rule $f : T \to \Delta A$ and a payment rule $(p_1, ..., p_n)$ are DSIC iff

1. $f$ is non-decreasing, and
2. payments are given by $p_i(t_i, t_i) = p_i(0, t_i) + t_i f_i(t_i, t_i) - \int_0^{t_i} f_i(x, t_i) dx$.

$\forall t_i \in T_i$, $\forall t'_i \in T_i$, $\forall i \in N$.

**Remark**: difference with the Roberts' theorem: Roberts' result gives a functional form, while Myerson's result is a more implicit property. Sometimes function forms help answering questions in a more direct manner.

**Proof**: ($\Rightarrow$) given that $(f, p)$ is DSIC.

Utility of agent $i$.

$u_i(t_i, t_i) = t_i f_i(t_i, t_i) - p_i(t_i, t_i)$, and $u_i(s_i, t_i) = s_i f_i(s_i, t_i) - p_i(s_i, t_i)$.

Since $(f, p)$ is DSIC,
\[ u_i(t_i, t_i) = t_i f(t_i, t_i) - p_i(t_i, t_i) \]
\[ > t_i f(a_i, t_i) - p_i(a_i, t_i) \]
\[ = \lambda_i f_i(a_i, t_i) + f_i(a_i, t_i)(t_i - \lambda_i) - p_i(a_i, t_i) \]
\[ = u_i(a_i, t_i) + f_i(a_i, t_i)(t_i - \lambda_i) \quad \text{--- (1)} \]

**Firing** \( t_i \), define \( g(t_i) = u_i(t_i, t_i), \phi(t_i) = f_i(t_i, t_i) \).

Hence, Eq. (1) can be written as

\[ g(t_i) > g(a_i) + \phi(a_i)(t_i - \lambda_i) \]

\[ \Rightarrow \phi(a_i) \text{ is a subgradient of } g \text{ at } a_i. \quad \text{--- (2)} \]

Next, need to show: \( g \) is convex.

pick \( z_i, \tilde{z}_i \in T_i \), define \( y_i = \lambda z_i + (1-\lambda) \tilde{z}_i, \lambda \in [0,1] \).

DC1 implies

\[ g(z_i) > g(y_i) + \phi(y_i)(z_i - y_i), \quad \text{and} \]
\[ g(\tilde{z}_i) > g(y_i) + \phi(y_i)(\tilde{z}_i - y_i) \]

\[ \Rightarrow \lambda g(z_i) + (1-\lambda) g(\tilde{z}_i) > g(y_i) + \phi(y_i) [\lambda z_i + (1-\lambda) \tilde{z}_i - y_i] \]

\[ = g(\lambda z_i + (1-\lambda) \tilde{z}_i) = 0 \]

\[ \Rightarrow g \text{ is convex.} \quad \text{--- (3)} \]

**Apply Lemmas 3 and 4**

Lemma 3 \( \Rightarrow \phi \) is non-decreasing, i.e., \( f_i(\cdot, t_i) \) is non-decreasing

\[ \Rightarrow \text{Part (1) is proved.} \]

Lemma 4 \( \Rightarrow g(t_i) = g(0) + \int_0^{t_i} \phi(\tau) d\tau \)
\[ U_i(t_i, t_i) = U_i(0, t_i) + \int_0^{t_i} f_i(x, t_i) \, dx \]
\[ t_i f_i(t_i, t_i) - p_i(t_i, t_i) = -p_i(0, t_i) + \int_0^{t_i} f_i(x, t_i) \, dx \]
\[ p_i(t_i, t_i) = p_i(0, t_i) + t_i f_i(t_i, t_i) - \int_0^{t_i} f_i(x, t_i) \, dx . \]

(\Leftarrow) Given: \( f \) is non-decreasing and payment formula.

\[
\text{proof by pictures} - \text{ assume } p_i(0, t_i) = 0
\]

\[
\left[ t_i f_i(t_i, t_i) - p_i(t_i, t_i) \right] - \left[ t_i f_i(s_i, t_i) - p_i(s_i, t_i) \right]
= (s_i - t_i) f_i(s_i, t_i) + \int_{s_i}^{t_i} f_i(x, t_i) \, dx > 0
\]

**Corollary:** An allocation rule in single object allocation setting is implementable in dominant strategies if it is non-decreasing.