Nash theorem and its proof

Theorem 1 (Nash (1951)) *Every finite game has a (mixed) Nash equilibrium.*

Proof: Define simplex to be

$$\Delta_k = \{x \in \mathbb{R}^{k+1}_{\geq 0} : \Sigma_{i=1}^{k+1} x_i = 1\}.$$

Clearly, this is a convex and compact set in $\mathbb{R}^{k+1}$. Consider two players (the case with n players is an extension of this idea). Say, player 1 has m strategies labeled $1, \ldots, m$ and player 2 has n strategies labeled $1, \ldots, n$. So, player 1's mixed strategy is a point in Δ_{m-1} and player 2's mixed strategy is a point in Δ_{n-1}. The set of mixed strategy profiles is a point in $\Delta_{m-1} \times \Delta_{n-1}$. Since we are in a two player game, the utilities can be expressed in terms of two matrices A and B, both in $\mathbb{R}^{m \times n}$, denoting the utilities of players 1 and 2 respectively at the pure strategy profiles given by the rows and columns of the matrices. For mixed strategies $p \in \Delta_{m-1}$ and $q \in \Delta_{n-1}$ for players 1 and 2 respectively

$$u_1(p,q) = p^\top A q, u_2(p,q) = p^\top B q.$$

Define the following quantities,

$$c_i(p,q) = \max\{A_i q - p^\top A q, 0\}, \text{ where } A_i \text{ is the } i^{th} \text{ row of } A, \ i \in \{1, \ldots, m\}.$$
$$d_j(p,q) = \max\{p^\top B_j - p^\top B q, 0\}, \text{where } B_j \text{ is the } j^{th} \text{ column of } B, \ j \in \{1, \ldots, n\}.$$

Clearly, both quantities are non-negative for all i, j.

Now, we define two functions P and Q as follows

$$P_i(p,q) = \frac{p_i + c_i(p,q)}{1 + \sum_{k=1}^m c_k(p,q)}, \ i \in \{1, \ldots, m\};$$
$$Q_j(p,q) = \frac{q_j + d_j(p,q)}{1 + \sum_{k=1}^n d_k(p,q)}, \ j \in \{1, \ldots, n\}.$$

Clearly, $P_i(p,q) \geq 0, \forall i$ and $\sum_{i=1}^m P_i(p,q) = 1$. Hence $P(p,q) \in \Delta_{m-1}$ and similarly we see that $Q(p,q) \in \Delta_{n-1}$. Define the transformation function

$$T(p,q) = (P(p,q), Q(p,q)).$$

We see that, $T : \Delta_{m-1} \times \Delta_{n-1} \mapsto \Delta_{m-1} \times \Delta_{n-1}$, and maps a convex and compact set onto itself. From the definitions it is clear that c_i and d_j's are continuous in (p,q), therefore, P_i's and Q_j's are also continuous which implies that T is continuous. Hence, by Brouwer's fixed point theorem,

$$\exists \ (p^*, q^*) \text{ s.t. } T(p^*, q^*) = (p^*, q^*).$$

Claim 2

$$\sum_{k=1}^{m} c_k(p^*, q^*) = 0; \qquad \sum_{k=1}^{n} d_k(p^*, q^*) = 0.$$

Proof:[of Claim] Suppose the claim is false, i.e., $\sum_{k=1}^{m} c_k(p^*, q^*) > 0$. Since (p^*, q^*) is a fixed point of T

$$p_i^* = \frac{p_i^* + c_i(p^*, q^*)}{1 + \sum_{k=1}^{m} c_k(p^*, q^*)} \Rightarrow p_i^* \left(\sum_{k=1}^{m} c_k(p^*, q^*) \right) = c_i(p^*, q^*). \tag{1}$$

Define a subset of indices as $I = \{i : p_i^* > 0\}$. We see that

$$I = \{i : p_i^* > 0\} = \{i : c_i(p^*, q^*) > 0\} = \{i : A_i q^* > p^{*\top} A q^*\}. \tag{2}$$

The first equality follows from eq. (1) and our assumption that $\sum_{k=1}^{m} c_k(p, q) > 0$. The second equality come from the definition of c_i. Define $u_i^* := p^{*\top} A q^*$.

Now we see

$$u_1^* = \sum_{i=1}^{m} p_i^* A_i q^* = \sum_{i \in I} p_i^* (A_i q^*) > \left(\sum_{i \in I} p_i^* \right) u_1^* = u_1^*.$$

The first equality is by definition, the second inequality holds since p_i^* is positive only on I (by definition), the inequality holds from eq. (2), and the last equality holds since u_i^* is a scalar and comes out of the summation. The inequality above is a contradiction. Similarly we can prove the claim for $\sum_k d_k$. Hence our claim is proved. ∎

By this claim, $\sum_{k=1}^{m} c_k(p^*, q^*) = 0$. Since $c_k(p^*, q^*) \geq 0, \forall k = 1, \ldots, m$, it implies that $c_k(p^*, q^*) = 0 \forall k = 1, \ldots, m$. By definition of c_i's, we then have

$$\begin{aligned} A_i q* &\leq p^{*\top} A q^* \\ \Rightarrow \sum_{i=1}^{m} p_i' A_i q^* &\leq p^{*\top} A q^*. \end{aligned}$$

The implication holds for any arbitrary mixed strategy p' of player 1. Similarly we can show that q^* is a best response for player 2 against the mixed strategy p^* played by player 1. Therefore (p^*, q^*) is a MSNE. ∎