CS 405/6001: Game Theory and Algorithmic Mechanism Design	**Jul–Nov 2026**

Lecture 3: Dominance and Nash Equilibrium

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Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor.*

3.1 Recap

Last week we compressed a game into its **strategic form** $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$, and fixed the notation s_{-i} for "everyone except i." We saw several strategic-form games but had no systematic way to say what a rational player *will* do. Today we build the first two solution concepts of the course: **domination** and **Nash equilibrium**.

Throughout, we will meet a running pricing duel between two streaming platforms, a sealed-bid auction for an indivisible block of shared compute, and a sport-selection game between two friends.

3.2 Domination

Definition 3.1 (Strictly dominated strategy, MSZ Def. 4.6) *A strategy $s_i' \in S_i$ of player i is **strictly dominated** if there exists $s_i \in S_i$ such that $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$ for every $s_{-i} \in S_{-i}$. We say s_i **strictly dominates** s_i'.*

Definition 3.2 (Weakly dominated strategy, MSZ Def. 4.12) *A strategy s_i' is **weakly dominated** by s_i if $u_i(s_i, s_{-i}) \geqslant u_i(s_i', s_{-i})$ for every $s_{-i} \in S_{-i}$, with strict inequality for at least one $\tilde{s}_{-i}$. This $\tilde{s}_{-i}$ is specific to the pair (s_i, s_i'): different pairs of strategies may need different witnesses $\tilde{s}_{-i}$.*

Example 3.3 (A pricing duel) *Two streaming platforms,* StreamCo *and* RegionalPlay*, simultaneously post a monthly subscription price of* Low*,* Mid*, or* High*. Payoffs (in Rs. lakh of monthly profit) are:*

		RegionalPlay		
		Low	*Mid*	*High*
StreamCo	*Low*	$0,0$	$5,1$	$6,0$
	Mid	$1,5$	$6,6$	$8,2$
	High	$0,6$	$2,8$	$4,4$

For StreamCo, Mid *gives $(1,6,8)$ against RegionalPlay's three prices, strictly more than* Low*'s $(0,5,6)$ in* every *column ($1>0$, $6>5$, $8>6$), and strictly more than* High*'s $(0,2,4)$ in every column too. So* Mid *is already a **strictly dominant strategy** for StreamCo – no need to eliminate anything first. The matrix is symmetric, so by the identical argument* Mid *is strictly dominant for RegionalPlay as well.*

Example 3.4 (Spotting strict and weak dominance together) *Consider the two-player game*

		Player 2: L	C	R
Player 1	U	$1, 0$	$1, 3$	$3, 2$
	D	$-1, 6$	$0, 5$	$3, 3$

*For Player 1: comparing U to D column by column gives $1 > -1$, $1 > 0$, $3 = 3$ – U is at least as good as D everywhere and strictly better twice, so D is **weakly** (not strictly, because of the tie in column R) dominated by U. For Player 2: comparing C to R row by row gives $3 > 2$ (row U) and $5 > 3$ (row D) – strictly better both times, so R is **strictly** dominated by C. This pair of examples is worth keeping side by side: a single tied column is the entire difference between strict and weak domination.*

Definition 3.5 (Dominant strategy) *s_i is a strictly (weakly) **dominant strategy** for i if it strictly (weakly) dominates every other $s_i' \in S_i \setminus \{s_i\}$.*

Definition 3.6 (Dominant strategy equilibrium) *A profile $(s_1^*, \ldots, s_n^*)$ is a strict (weak) **dominant strategy equilibrium**, SDSE (WDSE), if s_i^* is a strict (weak) dominant strategy for every $i \in N$.*

In Example 3.3, Mid is a strictly dominant strategy for both platforms, so (Mid, Mid), with payoff $(6, 6)$, is the game's SDSE.

Example 3.7 (A weak dominant-strategy equilibrium can hide a better outcome) *Consider the two-player game*

		Player 2: D	E
Player 1	A	$5, 5$	$0, 5$
	B	$5, 0$	$1, 1$
	C	$4, 0$	$1, 1$

For Player 1: B ties A in column D ($5 = 5$) but beats it in column E ($1 > 0$), and beats C in column D ($5 > 4$) while tying it in column E ($1 = 1$) – so B weakly *dominates both A and C, making B Player 1's weakly dominant strategy. Symmetrically, E weakly dominates D for Player 2 (tied at row A, strictly better at rows B and C). So the WDSE is (B, E), with payoff $(1, 1)$.*

But look at (A, D): payoff $(5, 5)$, strictly better for both *players than the WDSE payoff. Is it an equilibrium too? At (A, D), Player 1 switching to B gives the same 5 (no improvement), and switching to C gives $4 < 5$ (worse) – no profitable deviation. Symmetrically for Player 2. So (A, D) is* also *a PSNE (a non-strict one, since some deviations are merely tied rather than strictly worse), and it is better for everyone than the equilibrium reached by weak dominance. Keep this tension in mind – we will show next lecture exactly why eliminating weakly dominated strategies can discard a superior equilibrium like this one, while eliminating strictly dominated strategies never can.*

3.2.1 A Weakly Dominant Strategy: Truthful Bidding for Shared Compute

Recall the GPU-credit budget from Lecture 1's divide-and-choose example. Suppose instead two labmates, Riya and Karan, are competing for a single *indivisible* block of GPU-hours that only one of them can use this week. Riya values it at v_1 and Karan at v_2 (private, known only to themselves). Each submits a sealed bid $b_i \in [0, M]$; the higher bidder gets the block and pays the *other player's* bid (ties go to Riya). This is a sealed-bid **second-price auction**.

$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & b_1 \geqslant b_2 \\ 0 & \text{otherwise} \end{cases} \qquad u_2(b_1, b_2) = \begin{cases} v_2 - b_1 & b_1 < b_2 \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

Theorem 3.8 (MSZ Thm. 4.15) *Bidding $b_i = v_i$ weakly dominates every other bid of player i.*

Proof:[Sketch] Fix $i = 1$ and let $B_{-1} := b_2$ be the rival's bid (the argument for $i = 2$ is symmetric). Suppose $b_1 < v_1$.

- If $B_{-1} \leqslant b_1$ or $B_{-1} > v_1$, bidding b_1 and bidding v_1 lead to the same outcome (both win, or both lose), so the two bids give identical utility.
- If $B_{-1} \in (b_1, v_1]$, bidding v_1 wins and yields $v_1 - B_{-1} \geqslant 0$, while bidding b_1 loses and yields 0 – strictly worse whenever $B_{-1} < v_1$.

So v_1 is never worse, and is sometimes strictly better, than $b_1 < v_1$. The case $b_1 > v_1$ is symmetric: it can only help win auctions Riya would rather have lost (paying more than her value). Hence $b_1 = v_1$ weakly dominates every other bid. ■

This result – that truth-telling is a dominant strategy in a second-price auction – is due to Vickrey [Vic61], and it is the seed from which the entire second half of this course (mechanism design, VCG) will grow; keep this proof technique in your pocket. Modern compute marketplaces (spot GPU markets, decentralized rendering networks) that use second-price-style rules inherit exactly this incentive property; see Parkes and Seuken [PS22] for a computer-science-flavored treatment of such market designs.

3.2.2 Existence of Dominant Strategies Is Not Guaranteed

Example 3.9 (A sport-selection game) *Two friends must independently choose an activity for tonight,* CHESS *or* FOOTBALL*; they only have fun if they end up doing the same thing.*

		Friend 2	
		Chess	*Football*
Friend 1	*Chess*	$2, 1$	$0, 0$
	Football	$0, 0$	$1, 2$

Neither CHESS *nor* FOOTBALL *dominates the other for either player: which is better depends entirely on what the other friend picks. No dominant strategy exists for anyone here.*

Remark 3.10 *Whenever dominance cannot explain a reasonable outcome – as in Example 3.9, where "both play Chess" and "both play Football" both look like sensible things friends would settle into – we need a strictly weaker, more broadly applicable solution concept. That is exactly the gap Nash equilibrium fills.*

3.3 Nash Equilibrium

The guiding idea, due to Nash [Nas51], is simple to state even before any formal definition: **no player gains by a unilateral deviation**.

Definition 3.11 (Best response) *s_i is a **best response** to s_{-i} if $u_i(s_i, s_{-i}) \geqslant u_i(s_i', s_{-i})$ for every $s_i' \in S_i$. Write $B_i(s_{-i})$ for the (possibly non-unique) set of best responses.*

Definition 3.12 (Pure strategy Nash equilibrium, MSZ Def. 4.17) *(s_i^*, s_{-i}^*) is a **PSNE** if $s_i^* \in B_i(s_{-i}^*)$ for every $i \in N$: no player gains by unilaterally deviating.*

In Example 3.9, both (CHESS, CHESS) and (FOOTBALL, FOOTBALL) are PSNE – neither friend wants to switch alone from either. Two equilibria, unequal payoffs. Deciding *which* equilibrium real players converge on without communicating is the subject of Schelling's focal points [Sch60]; we will not resolve that question formally, but it is worth remembering that Nash equilibrium alone does not always pin down a unique prediction.

Theorem 3.13 *SDSE $\implies$ WDSE $\implies$ PSNE.*

Proof: Strict domination of all alternatives is a stronger requirement than weak domination of all alternatives, giving the first implication. For the second: if s_i^* weakly dominates every $s_i' \neq s_i^*$, then in particular $u_i(s_i^*, s_{-i}^*) \geqslant u_i(s_i', s_{-i}^*)$ for every s_i' and every fixed s_{-i}^* – which is exactly the statement that s_i^* is a best response to s_{-i}^*. Applying this to every player gives a PSNE. ■

The converse fails in both places: Example 3.9, the sport-selection game, has PSNE but no dominant strategy of either strength.

3.3.1 Finding Equilibria by Elimination – and a Warning About Order

Example 3.14 (Elimination order can change the outcome) *Two players face the same 3×3 game; Player 1 picks a row, Player 2 picks a column.*

		Player 2		
		L	C	R
	T	$1,2$	$2,3$	$0,3$
Player 1	M	$2,2$	$2,1$	$3,2$
	B	$2,1$	$0,0$	$1,0$

Eliminating in the order T, R, B, C leaves (M, L) with payoff $(2,2)$. Eliminating in the order B, L, C, T leaves (M, R) with payoff $(3,2)$. Both *survive as candidates depending on elimination order – a fact we will pin down precisely (and show cannot happen for* strict *domination) once we study iterated elimination formally next lecture.*

3.4 A First Look at Risk: When Rationality of the Opponent Is in Doubt

Nash equilibrium assumes every player trusts that everyone else is rational *and* will play their part of the equilibrium. But what should a player do if they cannot be sure of that? If the opponent might not play their equilibrium action at all – through error, different information, or simply different incentives than we assumed – an equilibrium strategy can turn out badly.

This motivates asking a different question: forgetting about what the opponent is likely to do, what can a player *guarantee* for themselves in the worst case? That is the **maxmin** or **security** value, which we develop fully next lecture, where we will also meet two-player zero-sum games and see a classic example – a penalty shoot-out – where maxmin and minmax genuinely disagree, the first real motivation in this course for randomized (mixed) strategies.

3.5 Summary and Preview

We defined strict and weak domination, dominant strategies, and dominant-strategy equilibria; saw a game where dominance alone pins down the SDSE directly, and another where the WDSE hides a strictly better, non-strict PSNE; proved that truthful bidding is weakly dominant in a second-price auction (Vickrey [Vic61]); saw that dominant strategies need not exist; introduced Nash equilibrium as a strictly more general stability notion (Theorem 3.13); and flagged that the order of iterated elimination can matter. Next lecture: the maxmin value, its relationship to dominance and to PSNE, iterated elimination's effect on security levels and on the set of equilibria, and two-player zero-sum (matrix) games.

References

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[PS22] D. C. Parkes and S. Seuken, *Economics and Computation*, draft textbook, 2022 edition, Chapter 9 (Auctions).

[Sch60] T. C. Schelling, *The Strategy of Conflict*, Harvard University Press, 1960.

[Vic61] W. Vickrey, "Counterspeculation, auctions, and competitive sealed tenders," *Journal of Finance*, 16(1), 1961, pp. 8–37.