CS 405/6001: Game Theory and Algorithmic Mechanism Design **Jul–Nov 2026**

Lecture 17: The Median Voter Theorem and Task Sharing

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17.1 Recap

Last lecture we restricted the domain to **single-peaked** preferences. The alternatives lie on a line with a common order $<$. Each agent has a peak. Moving away from the peak, on either side, makes the agent worse off.

Our running example was a GPU cluster shared by several teams. The teams must agree on one power cap. Each team has an ideal cap. A cap that is too low slows training. A cap that is too high risks throttling and costs money. So each team's preference over caps is single-peaked.

We defined a **median voter SCF**. It fixes $n-1$ **phantom peaks** $B = \{y_1, \ldots, y_{n-1}\}$. These do not depend on the profile. For every profile P with reported peaks $p_1, \ldots, p_n$,

$$f(P) = \text{median}\big(y_1, \ldots, y_{n-1},\, p_1, \ldots, p_n\big),$$

where the median is taken with respect to $<$. We proved that every median voter SCF is strategyproof. We also showed that, for an SP rule on this domain, ONTO, unanimity, and Pareto efficiency are equivalent. Finally we defined anonymity: $f(P^\sigma) = f(P)$ for every permutation σ of the agents. Dictatorships are not anonymous.

Today we prove the characterization, and then move to a new domain where agents share a divisible task.

Theorem 17.1 (Moulin 1980) *Let $f : \mathcal{S}^n \to A$ be a strategyproof SCF on the single-peaked domain. Then f is ONTO and anonymous (ANON) if and only if f is a median voter SCF.*

17.2 The Easy Direction

Let f be a median voter SCF with phantoms B.

- **SP.** Suppose $f(P) = a$ and team i has peak $p_i < a$. To move the median, team i must report a peak on the right of a. That can only move the outcome to the right, further from p_i. The case $p_i > a$ is symmetric.
- **ANON.** Permuting the teams does not change the multiset of peaks. So the median does not change.
- **ONTO.** Fix any $a \in A$. Let every team report peak a. Among the $2n-1$ points, n are at a. That is a strict majority, so the median is a, wherever the phantoms are.

The real content of Theorem 17.1 is the other direction. We are handed an arbitrary f that is SP, ANON, and ONTO. We must *find* phantoms that make f a median rule.

17.3 Constructing the Phantom Peaks

Define two extreme preferences for each team i:

- P_i^0: the peak is the **leftmost** alternative. Under P_i^0, a smaller alternative is always better.
- P_i^1: the peak is the **rightmost** alternative. Under P_i^1, a larger alternative is always better.

Both are single-peaked, so both are feasible reports. The proof is constructive. We build the phantoms from f itself:

$$y_j = f\big(\underbrace{P_1^0, \ldots, P_{n-j}^0}_{n-j \text{ leftmost}}, \underbrace{P_{n-j+1}^1, \ldots, P_n^1}_{j \text{ rightmost}}\big), \qquad j = 1, \ldots, n-1. \tag{17.1}$$

In words, y_j is what f picks when j teams ask for the highest cap and the rest ask for the lowest. By anonymity, it does not matter *which* teams go right. Only the count j matters.

Claim 17.2 *The phantoms are sorted: $y_j \leqslant y_{j+1}$ for $j = 1, \ldots, n-2$.*

Proof: Compare the two defining profiles:

$$y_j = f(P_1^0, \ldots, P_{n-j-1}^0,\ P_{n-j}^0,\ P_{n-j+1}^1, \ldots, P_n^1),$$
$$y_{j+1} = f(P_1^0, \ldots, P_{n-j-1}^0,\ P_{n-j}^1,\ P_{n-j+1}^1, \ldots, P_n^1).$$

They differ only in team $n-j$'s report. Suppose its true preference is P_{n-j}^0. By SP, reporting P_{n-j}^1 does not help, so $y_j\ R_{n-j}^0\ y_{j+1}$. Under P_{n-j}^0 smaller is better. Hence $y_j \leqslant y_{j+1}$. ■

Now define the candidate rule

$$g(P) = \text{median}\big(y_1, \ldots, y_{n-1}, p_1, \ldots, p_n\big).$$

We show $f(P) = g(P)$ for every profile P.

Fix P. By ANON, relabel the teams so that $p_1 \leqslant p_2 \leqslant \cdots \leqslant p_n$. Let $a = g(P)$. There are two cases: a is a phantom peak, or a is a team's peak.

17.4 Case 1: The Median Is a Phantom

Let $a = y_j$ for some $j \in \{1, \ldots, n-1\}$. The median of $2n-1$ points has exactly $n-1$ points on each side. By Claim 17.2, $j-1$ phantoms lie to the left of y_j. So $(n-1)-(j-1) = n-j$ team peaks lie to the left too. Therefore

$$p_1 \leqslant \cdots \leqslant p_{n-j} \leqslant y_j \leqslant p_{n-j+1} \leqslant \cdots \leqslant p_n. \tag{17.2}$$

Walking the left teams in. Start from the profile that defines y_j in (17.1), and replace team 1's report by its true preference:

$$f(P_1, P_2^0, \ldots, P_{n-j}^0, P_{n-j+1}^1, \ldots, P_n^1) = b \quad \text{(say)}.$$

- If team 1's true preference were P_1^0, SP gives $y_j\ R_1^0\ b$. Since P_1^0 peaks at the far left, $y_j \leqslant b$.
- If team 1's true preference is P_1, SP gives $b\ R_1\ y_j$. From (17.2), $p_1 \leqslant y_j \leqslant b$. All three points are on the same side of p_1, so single-peakedness gives $b \leqslant y_j$.

Hence $b = y_j$. Every team $k \leqslant n - j$ also has $p_k \leqslant y_j$, so the same two lines work for teams $2, \ldots, n - j$ in turn:

$$f(P_1, \ldots, P_{n-j}, P^1_{n-j+1}, \ldots, P^1_n) = y_j.$$

Walking the right teams in. Now replace team n's report:

$$f(P_1, \ldots, P_{n-j}, P^1_{n-j+1}, \ldots, P^1_{n-1}, P_n) = b \quad \text{(say)}.$$

SP for true type P^1_n gives $y_j \; R^1_n \; b$, so $b \leqslant y_j$. SP for true type P_n gives $b \; R_n \; y_j$. Since $b \leqslant y_j \leqslant p_n$, single-peakedness gives $y_j \leqslant b$. So $b = y_j$. Repeat for teams $n - 1, \ldots, n - j + 1$ to obtain

$$f(P_1, \ldots, P_n) = y_j = g(P).$$

Each switch was pinned between two SP inequalities. No team could move the outcome, so the phantom survives all the way to the true profile.

17.5 Case 2: The Median Is a Team's Peak

We prove this case for two teams, $N = \{1, 2\}$. There is one phantom y_1 and $g(P) = \text{median}(p_1, p_2, y_1)$. The general case repeats the same ideas with more bookkeeping [Mou80].

Recall from last lecture that, under SP, ONTO is equivalent to Pareto efficiency. On this domain, PE means $f(P)$ lies between the smallest and the largest peak.

Lemma 17.3 *Let $N = \{1, 2\}$. If P and P' have the same peaks, $P_i(1) = P'_i(1)$ for $i = 1, 2$, then $f(P) = f(P')$.*

Proof: Let $a = P_1(1) = P'_1(1)$ and $b = P_2(1) = P'_2(1)$. Let $x = f(P_1, P_2)$ and $y = f(P'_1, P_2)$. By SP, $x \; R_1 \; y$ and $y \; R'_1 \; x$.

If x and y are on the same side of a, both R_1 and R'_1 prefer the one closer to a. The two inequalities then force $x = y$.

Otherwise x and y lie on opposite sides of a, say $x < a < y$. By PE, both x and y must lie in $[\min(a, b), \max(a, b)]$, since both profiles have peaks a and b. But an interval with endpoint a cannot contain points on both sides of a. Contradiction. So $f(P_1, P_2) = f(P'_1, P_2)$. The same argument applied to team 2 gives $f(P'_1, P_2) = f(P'_1, P'_2)$. ∎

Now take a profile P with $p_1 = a$, $p_2 = b$. By assumption the median of $\{a, b, y_1\}$ is a team's peak. By ANON, assume it is a. Suppose, for contradiction, $f(P) = c \neq a$. By PE, c lies between a and b. Since a is the median, there are two orderings.

Case 2.1: $b < a < y_1$**.** By PE, $c < a$. So c and y_1 lie on opposite sides of a.

- Construct P'_1 with $P'_1(1) = a$ that ranks y_1 above c. This is possible because the two are on different sides of a.
- By Lemma 17.3, $f(P'_1, P_2) = c$.
- Now team 1 reports P^1_1, whose peak is the rightmost alternative. Then $b < y_1 \leqslant P^1_1(1)$, so the median of $\{b, y_1, P^1_1(1)\}$ is the phantom y_1. By Case 1, $f(P^1_1, P_2) = y_1$.

- Team 1 with true preference P_1' gets c by being truthful and y_1 by reporting P_1^1. It prefers y_1. This contradicts SP.

Case 2.2: $y_1 < a < b$**.** This is the mirror image. By PE, $a < c$. Construct P_1' with peak a that ranks y_1 above c. By Lemma 17.3, $f(P_1', P_2) = c$. Team 1 reports P_1^0. Then $P_1^0(1) \leqslant y_1 < b$, the median is the phantom y_1, and Case 1 gives $f(P_1^0, P_2) = y_1$. Team 1 with P_1' gains by lying. Contradiction.

So $f(P) = a = g(P)$ in Case 2 as well. This completes the proof of Theorem 17.1 for two agents.

Remark 17.4 *Gibbard–Satterthwaite says that on the unrestricted domain, SP and ONTO (with at least three alternatives) leave only dictatorships. Theorem 17.1 says that on the single-peaked domain, adding ANON leaves a whole family: one rule for each placement of the $n-1$ phantoms. Putting all phantoms at the far left gives the leftmost-peak rule. Putting them all at the far right gives the rightmost-peak rule. Splitting them evenly gives the plain median of the team peaks (for odd n).*

17.6 Sharing a Divisible Task

We now change the setting. A gig platform has **one unit** of delivery load tonight. It splits the load among n riders. Rider i gets a share $s_i \in [0, 1]$, and $\sum_{i \in N} s_i = 1$. Every rider has a most preferred share.

Why a peak? Suppose rider i is paid w per unit of load, so working t_i earns $w\, t_i$. Fatigue costs $c_i\, t_i^2$, since late orders hurt more than early ones. The net payoff is

$$w\, t_i - c_i\, t_i^2,$$

which is maximized at $t_i = w/2c_i$ and decreases monotonically on both sides. So each rider's preference is single-peaked in her own share.

Single-peaked over shares, not over allocations. An alternative is now a whole allocation $(s_1, \ldots, s_n)$. Rider 1 only looks at s_1. For example, rider 1 likes $(0.2, 0.4, 0.4)$ and $(0.2, 0.6, 0.2)$ equally. With three riders the set of allocations is a two-dimensional simplex. There is no single common order on allocations that makes every rider's preference single-peaked. So Theorem 17.1 does not apply directly, and we need new rules.

Denote the domain of task allocation preferences by T and the set of allocations by A. An **SCF** is $f : T^n \to A$. For $P \in T^n$, $f(P) = (f_1(P), \ldots, f_n(P))$ with $f_i(P) \in [0, 1]$ and $\sum_{i \in N} f_i(P) = 1$. Rider i's peak share is p_i.

Definition 17.5 (Pareto efficiency) *f is **Pareto efficient** (PE) if there is no profile P and allocation $a \in A$ such that a R_i $f(P)$ for all $i \in N$ and a P_j $f(P)$ for some $j \in N$.*

Theorem 17.6 *An allocation $(s_1, \ldots, s_n)$ at profile P is Pareto efficient if and only if:*

1. *when $\sum_i p_i = 1$: $s_i = p_i$ for all i (the unique PE allocation);*
2. *when $\sum_i p_i > 1$: $s_i \leqslant p_i$ for all i;*
3. *when $\sum_i p_i < 1$: $s_i \geqslant p_i$ for all i.*

Proof:[Proof sketch] *Necessity.* If $\sum_i p_i = 1$ and some $s_i \neq p_i$, give everyone her peak. Nobody is worse off and rider i is strictly better off. If $\sum_i p_i > 1$ and some rider j has $s_j > p_j$, then since shares sum to

$1 < \sum_i p_i$, some rider k has $s_k < p_k$. Move a small amount of load from j to k. Rider j wanted less and rider k wanted more, so both strictly gain, and nobody else is affected. The case $\sum_i p_i < 1$ is symmetric.

Sufficiency. Say $\sum_i p_i > 1$ and $s_i \leqslant p_i$ for all i. Any other feasible allocation must give some rider k strictly less than $s_k \leqslant p_k$, which makes k strictly worse off. The other cases are similar. ■

In short, PE means that **every rider stays on the same side of her peak**. Next lecture we ask which rule is PE, anonymous, and strategyproof at the same time.

17.7 Summary

The phantoms of an SP, ANON, ONTO rule are read off from f itself: y_j is the outcome when j agents report the rightmost peak. SP forces them to be sorted. When the median is a phantom, two SP inequalities pin every agent's switch, and the phantom survives. When the median is an agent's peak, an agent who could steer the outcome to the phantom would manipulate, so the agent's peak must win. On single-peaked domains, SP, ONTO, and ANON characterize exactly the median voter rules. In task sharing, preferences are single-peaked in one's own share, and PE means everyone stays on the same side of her peak.

References

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[Spr91] Y. Sprumont, "The division problem with single-peaked preferences: a characterization of the uniform allocation rule," *Econometrica*, 59(2), 1991, pp. 509–519.