CS 405/6001: Game Theory and Algorithmic Mechanism Design **Jul–Nov 2026**

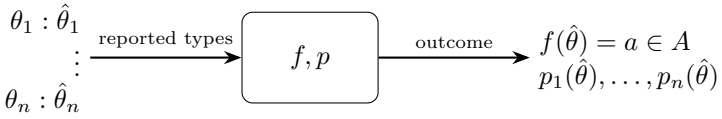
Lecture 19: The VCG Mechanism and Combinatorial Allocations

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Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor.*

19.1 Recap

Last week we let a mechanism move money. A mechanism is now a pair (f, p): an **allocation rule** $f : \Theta_1 \times \cdots \times \Theta_n \to A$ and a **payment function** $p_i : \Theta_1 \times \cdots \times \Theta_n \to \mathbb{R}$ for every agent. We recall only what today needs; the proofs are in Lecture 18.



Quasi-linear utility. Agent i's utility is $u_i(a, p_i, \theta_i) = v_i(a, \theta_i) - p_i(\theta)$. By convention, $p_i(\theta) > 0$ means agent i *pays*, and $p_i(\theta) < 0$ means agent i *receives* money, a **subsidy**. For instance, if your lab's GPU-cluster scheduler prints $p_3(\theta) = -6$ on the chargeback ledger, user 3 is credited 6 units.

Payment properties. A payment rule satisfies

- **no deficit** if $\sum_{i \in N} p_i(\theta) \geqslant 0$ for all θ: the mechanism needs no external funds;
- **no subsidy** if $p_i(\theta) \geqslant 0$ for all i and θ: no agent ever receives money;
- **budget balance** if $\sum_{i \in N} p_i(\theta) = 0$ for all θ: money only moves between agents.

No subsidy implies no deficit, since a sum of non-negative terms is non-negative. Budget balance also implies no deficit, since a sum equal to zero is non-negative. No other implication among the three holds in general; in particular, a budget-balanced rule can still pay some agents.

Efficiency and Groves. The allocatively efficient rule is $f^{AE}(\theta) \in \arg\max_{a \in A} \sum_{i \in N} v_i(a, \theta_i)$, and in the quasi-linear domain a mechanism is Pareto optimal if and only if $f = f^{AE}$. A **Groves mechanism** pairs f^{AE} with

$$p_i^G(\theta) = h_i(\theta_{-i}) - \sum_{j \neq i} v_j\big(f^{AE}(\theta), \theta_j\big),$$

and is DSIC for *any* choice of $h_i : \Theta_{-i} \to \mathbb{R}$. Truthfulness does not depend on h_i, but solvency does. Today we make the canonical choice of h_i.

19.2 VCG Mechanism

19.2.1 Definition

The Vickrey-Clarke-Groves (VCG) mechanism, also known as the **pivotal mechanism** [V61,C71,G73], is the most popular mechanism in the Groves class. It is given by the unique function

$$h_i(\theta_{-i}) = \max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j).$$

Definition 19.1 (VCG payment rule) *The VCG payment is*

$$p_i^{VCG}(\theta) = \max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j) - \sum_{j \neq i} v_j(f^{AE}(\theta), \theta_j).$$

19.2.2 Interpretation of Payment and Utility

The VCG payment is interpreted as the **externality**, or **marginal damage**, agent i imposes on the others:

$$p_i^{VCG}(\theta) = \text{Sum value of others (in absence of } i) - \text{Sum value of others (in presence of } i).$$

An agent is charged a payment if their presence changes the outcome (or the overall utility of others) in a way that harms others.

The **utility** of agent i under the VCG mechanism is

$$u_i^{VCG}(\theta) = v_i(f^{AE}(\theta), \theta_i) - p_i^{VCG}(\theta) = \sum_{j \in N} v_j(f^{AE}(\theta), \theta_j) - \max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j).$$

This utility is interpreted as the **marginal contribution** of i to the social welfare.

19.2.3 Examples

19.2.3.1 Single Object Allocation (Second-Price Auction)

Four classmates bid in sealed envelopes for one front-row ticket to the campus concert, with types $\theta = \{10, 8, 9, 5\}$.

- **Efficient allocation**: the ticket goes to bidder 1 ($\theta_1 = 10$).
- **VCG payments**: for the winner,

$$p_1^{VCG} = \max_{a} \sum_{j \neq 1} v_j(a, \theta_j) - \sum_{j \neq 1} v_j(f^{AE}(\theta), \theta_j) = 9 - 0 = 9.$$

A loser's absence changes nothing for the others, so each loser pays 0. The payments are $\{9, 0, 0, 0\}$: the highest bidder wins and pays the second highest bid. This is the **second-price auction**: VCG applied to a single object is exactly Vickrey's auction, and the winner pays the highest value among the others, which is precisely the welfare they denied the rest.

19.2.3.2 Public Project Choice (Pivotal Agent)

Three student bodies vote on one shared campus facility, with valuations for {Football (F), Library (L), Food Joint (J)}:

Agent	Football	Library	Food Joint	Total
A	0	70	50	
B	95	10	50	
C	10	50	50	
$\sum v_i(a, \theta_i)$	105	130	150	

VCG allocation: Food Joint (J), with total social welfare 150.

Payments (using $p_i^{VCG}(\theta) = \max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j) - \sum_{j \neq i} v_j(f^{AE}(\theta), \theta_j)$):

- **Player A**: $\max_a \sum_{j \neq A} v_j(a, \theta_j) = \max\{95 + 10,\ 10 + 50,\ 50 + 50\} = 105$. And $\sum_{j \neq A} v_j(f^{AE}(\theta), \theta_j) = 50 + 50 = 100$. So $p_A = 105 - 100 = 5$.
- **Player B**: $\max_a \sum_{j \neq B} v_j(a, \theta_j) = \max\{0 + 10,\ 70 + 50,\ 50 + 50\} = 120$. And $\sum_{j \neq B} v_j(f^{AE}(\theta), \theta_j) = 50 + 50 = 100$. So $p_B = 120 - 100 = 20$.
- **Player C**: $\max_a \sum_{j \neq C} v_j(a, \theta_j) = \max\{0 + 95,\ 70 + 10,\ 50 + 50\} = 100$. And $\sum_{j \neq C} v_j(f^{AE}(\theta), \theta_j) = 50 + 50 = 100$. So $p_C = 100 - 100 = 0$.

Player C is a **non-pivotal agent**, because the outcome (Food Joint) is still optimal without them. The agent whose presence changes the outcome is charged money. They are the **pivotal players**.

19.2.3.3 Combinatorial Allocation

Combinatorial allocation involves the sale of multiple objects where agents have valuations over bundles. Consider two ad inventory units, 1 and 2, auctioned to two bidders. The valuations over all possible combinations are:

Bundle (a)	$\emptyset$	$\{1\}$	$\{2\}$	$\{1, 2\}$
v_1 (θ_1)	0	8	6	12
v_2 (θ_2)	0	9	4	14

Efficient allocation: $\{1\}$ goes to player 2 and $\{2\}$ goes to player 1, with welfare $9 + 6 = 15$, beating the alternatives 12 (all to 1) and 14 (all to 2). Here type is the value itself, so we can also write $v_i(a, \theta_i) = \theta_i(a)$. The VCG payment is

$$p_i^{\text{VCG}}(\theta_1, \theta_2) = \max_{a \in A} \sum_{j \neq i} \theta_j(a) - \sum_{j \neq i} \theta_j(f^{\text{AE}}(\theta)).$$

$$p_1^{\text{VCG}}(\theta_1, \theta_2) = \max_{a \in A} \sum_{j \neq 1} \theta_j(a) - \sum_{j \neq 1} \theta_j(f^{\text{AE}}(\theta)) = 14 - 9 = 5; \qquad \text{payoff} = 6 - 5 = 1$$

$$p_2^{\text{VCG}}(\theta_1, \theta_2) = \max_{a \in A} \sum_{j \neq 2} \theta_j(a) - \sum_{j \neq 2} \theta_j(f^{\text{AE}}(\theta)) = 12 - 6 = 6; \qquad \text{payoff} = 9 - 6 = 3$$

Hence every agent gets a non-negative payoff – something we discuss properly in Section 19.4.

19.3 VCG Mechanism in the Combinatorial Allocation Setting

19.3.1 Notation

- $M = \{1, 2, \ldots, m\}$: set of objects.
- $\Omega = \{S : S \subseteq M\}$: set of bundles.
- $\theta_i : \Omega \to \mathbb{R}$: type/value of agent i. Hence $\theta_i(S)$ is the value of agent i for the bundle S.
- **Assumption**: we assume $\theta_i(S) \geqslant 0$ for all $S \in \Omega$ (every item is a 'good'; an item that gives negative valuation is called a 'bad'). Equivalently, $\theta_i(S \cup \{j\}) - \theta_i(S) \geqslant 0$ for every S with $j \notin S$.
- **Allocation** (a): $a = (a_0, a_1, \ldots, a_n)$, where $a_i \in \Omega$, $a_i \cap a_j = \emptyset$ if $i \neq j$, and $\cup_{i=0}^{n} a_i = M$. Here a_i is the bundle allocated to i and a_0 is the set of unallocated objects. A is the set of all such allocations. Assume $\theta_i(\emptyset) = 0$.
- We also assume **selfish valuations**: $\theta_i(a) = \theta_i(a_i)$, i.e., agent i's valuation does not depend on bundles of other agents.

This is the model behind every bundled-resource sale you are likely to meet in practice: spectrum licences, reserved cloud capacity, and the ad-slot packages a demand-side platform buys in one shot.

VCG mechanism has several useful properties:

1. It is DSIC.
2. It is efficient.
3. Payments are non-negative (hence no subsidy).

19.3.2 An Agent Who Gets Nothing Pays Nothing

Claim 19.2 *In the allocation of goods, the VCG payment for an agent that gets no object in the efficient allocation is zero.*

Proof: Let a^* be the efficient allocation (maximizing total social welfare), i.e., $a^* \in \arg\max_{a \in A} \sum_{j \in N} \theta_j(a)$. Assume agent i gets no object in this efficient allocation, so $a_i^* = \emptyset$, which implies $\theta_i(a^*) = \theta_i(a_i^*) = 0$.

Define a_{-i}^* as the allocation that maximizes the welfare of agents excluding i:

$$a_{-i}^* \in \arg\max_{b \in A_{-i}} \sum_{j \in N \setminus \{i\}} \theta_j(b)$$

The VCG payment formula is

$$p_i^{\mathrm{VCG}}(\theta) = \sum_{j \neq i} \theta_j(a_{-i}^*) - \sum_{j \neq i} \theta_j(a^*)$$

Step 1: $p_i^{\mathrm{VCG}}(\theta) \geqslant 0$. Since agent i gets nothing in a^*, the allocation a^* also splits the objects among $N \setminus \{i\}$ alone, i.e., $a^* \in A_{-i}$. As a_{-i}^* maximizes the others' welfare over A_{-i},

$$\sum_{j \neq i} \theta_j(a^*) \leqslant \sum_{j \neq i} \theta_j(a_{-i}^*), \quad \text{so} \quad p_i^{\mathrm{VCG}}(\theta) \geqslant 0.$$

This step genuinely needs $a^* \in A_{-i}$. In general the efficient allocation need not be available once i is removed, which is why VCG payments are not non-negative in every domain; here it is available because i receives nothing.

Step 2: $p_i^{\text{VCG}}(\theta) \leqslant 0$. Agent i receives nothing in either allocation: in a^*_{-i} because i is absent, and in a^* by assumption. Hence $\theta_i(a^*_{-i}) = 0$ and $\theta_i(a^*) = \theta_i(\emptyset) = 0$. Adding the first to the first sum and the second to the second sum changes nothing:

$$p_i^{\text{VCG}}(\theta) = \sum_{j \in N} \theta_j(a^*_{-i}) - \sum_{j \in N} \theta_j(a^*) \leqslant 0,$$

since $a^*_{-i} \in A_{-i} \subseteq A$ and a^* maximizes the full social welfare over A.

Combining the two steps, $p_i^{\text{VCG}}(\theta) = 0$. ■

Note where each hypothesis was used. Step 1 needs a^* to remain feasible without i, and Step 2 needs $\theta_i(\emptyset) = 0$ together with $A_{-i} \subseteq A$. Drop either one and the claim fails, which is exactly what happens in the cost-minimization setting of Lecture 20, where valuations are negative and payments run the other way.

19.4 Individual Rationality

DSIC says that once you are inside the mechanism you should tell the truth. It says nothing about whether you should enter at all. That is the job of individual rationality.

Definition 19.3 (Ex-post Individual Rationality) *A direct mechanism (f, p) is ex-post individually rational (IR) if for every possible profile of types $\theta \in \Theta$, every agent receives a non-negative utility from participating, i.e.,*

$$v_i(f(\theta), \theta_i) - p_i(\theta) \geqslant 0, \quad \forall i \in N.$$

This property ensures that by participating in the mechanism, no agent gets a negative utility. Hence a rational agent has incentive to participate in the game.

Two properties are sufficient to guarantee Ex-Post Individual Rationality.

Definition 19.4 (Choice Set Monotonicity) *For all $i \in N$, $A_{-i} \subseteq A$.*

Interpretation: the addition of agents weakly increases choices. The set of allocations available when agent i is absent (A_{-i}) must be a subset of the full set of allocations (A).

Definition 19.5 (No Negative Externality) *For all $i \in N$, $\theta \in \Theta$, $v_i(a^*_{-i}(\theta_{-i}), \theta_i) \geqslant 0$.*

Interpretation: the efficient allocation chosen without agent i (a^*_{-i}) must yield a non-negative value to agent i. This is typically met in domains involving the allocation of goods, where all values are naturally $\geqslant 0$.

Example 19.6 (Our combinatorial allocation domain satisfies both) *Any split of the objects among $N \setminus \{i\}$ is also a split among N in which i gets $\emptyset$, so $A_{-i} \subseteq A$: choice set monotonicity holds. In a^*_{-i} agent i gets nothing, so $v_i(a^*_{-i}, \theta_i) = \theta_i(\emptyset) = 0 \geqslant 0$: no negative externality holds. Concretely, in the two-item example, removing player 1 makes the best split hand $\{1, 2\}$ to player 2; that split lies in A, with player 1 holding $\emptyset$ and valuing it at 0.*

Claim 19.7 *If the allocations satisfy choice set monotonicity and the valuations have no negative externality, then the VCG mechanism is individually rational.*

Proof: We recall the definitions of the allocations based on the reported types θ:

$$a^*(\theta_i, \theta_{-i}) \in \arg\max_{a \in A} \sum_{k \in N} v_k(a, \theta_k) \quad \text{(Efficient Allocation)}$$

$$a^*_{-i}(\theta_{-i}) \in \arg\max_{a \in A_{-i}} \sum_{j \neq i} v_j(a, \theta_j) \quad \text{(Max SW Excluding } i\text{)}$$

Writing $f^{\text{AE}}(\theta_i, \theta_{-i}) = a^*(\theta_i, \theta_{-i})$ and substituting the VCG payment,

$$\begin{aligned}
\text{utility of player } i &= v_i(f^{\text{AE}}(\theta_i, \theta_{-i}), \theta_i) - p_i^{\text{VCG}}(\theta_i, \theta_{-i}) \\
\text{[by defn of VCG payment]} \quad &= \sum_{j \in N} v_j(a^*(\theta_i, \theta_{-i}), \theta_j) - \sum_{j \neq i} v_j(a^*_{-i}(\theta_{-i}), \theta_j) \\
\text{[choice set monotonicity]} \quad &\geqslant \sum_{j \in N} v_j(a^*_{-i}(\theta_{-i}), \theta_j) - \sum_{j \neq i} v_j(a^*_{-i}(\theta_{-i}), \theta_j) \\
&= v_i(a^*_{-i}(\theta_{-i}), \theta_i) \;\geqslant\; 0 \quad \text{[no negative externality]}
\end{aligned}$$

The inequality in the third line uses choice set monotonicity: it guarantees $a^*_{-i}(\theta_{-i}) \in A_{-i} \subseteq A$, so a^*, which maximizes the overall social welfare over A, is at least as good as a^*_{-i} on that same sum. The final step is the no negative externality condition. Hence $u_i(\theta) \geqslant 0$. ■

Corollary 19.8 *VCG is ex-post IR for combinatorial allocations.*

This settles the two non-negative payoffs we computed in the two-item example, and together with Claim 19.2 it explains the pattern in every example of this lecture: nobody is forced to lose money, and nobody is billed for an empty hand.

19.5 Summary

- The **VCG mechanism** is the Groves mechanism with $h_i(\theta_{-i}) = \max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j)$.
- Each agent pays the **externality** it imposes on the others, and its utility is its **marginal contribution** to social welfare.
- On a single object VCG is the **second price auction**. Only **pivotal** agents pay.
- In combinatorial allocations of goods, an agent who receives nothing pays nothing.
- VCG is ex-post individually rational under choice set monotonicity and no negative externality.

VCG Payment:

$$p_i^{VCG}(\theta) = \max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j) - \sum_{j \neq i} v_j(f^{AE}(\theta), \theta_j)$$

VCG Utility (Marginal Contribution):

$$u_i^{VCG}(\theta) = \sum_{j \in N} v_j(f^{AE}(\theta), \theta_j) - \max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j)$$

Preview. Next lecture we put VCG to work in sponsored search and position auctions, and then catalogue where it breaks: collusion, frugality, revenue monotonicity, and budget balance.

References

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