CS 405/6001: Game Theory and Algorithmic Mechanism Design **Jul–Nov 2026**

Lecture 23: Optimal Mechanism Design

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Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor.*

23.1 Recap and Plan

So far we designed mechanisms that are truthful. Now the designer also wants *money*. We ask: among all truthful and individually rational mechanisms, which one earns the most revenue on average?

A prior-free worst-case guarantee on revenue is usually out of reach. So we adopt a **Bayesian** view. Each agent's private type is drawn from a commonly known prior, and we compare mechanisms by their *expected* revenue. Incentives are therefore stated in interim terms, that is, in expectation over the other agents' types.

The lecture runs in four chunks:

1. One buyer: revenue becomes virtual value (Section 23.2).
2. MHR and the posted price (Section 23.3).
3. Many buyers: maximize virtual surplus (Section 23.4).
4. Myerson's optimal auction and an example (Section 23.5).

23.1.1 Bayesian set-up

There are n agents. Agent i has a private type $t_i \in T_i = [0, b_i]$. The type profile is $t = (t_1, \ldots, t_n) \in T = \times_{i=1}^{n} T_i$, drawn from a common prior G with density g. A mechanism $(f, p_1, \ldots, p_n)$ has allocation f and payments p_i. The interim allocation and payment of agent i, when she reports s_i and her true type is t_i, are

$$\alpha_i(s_i; t_i) = \int_{T_{-i}} f_i(s_i, s_{-i})\, g_{-i}(s_{-i} \mid t_i)\, ds_{-i}, \qquad \pi_i(s_i; t_i) = \int_{T_{-i}} p_i(s_i, s_{-i})\, g_{-i}(s_{-i} \mid t_i)\, ds_{-i}.$$

Her expected utility is $u_i(s_i; t_i) = t_i\, \alpha_i(s_i; t_i) - \pi_i(s_i; t_i)$.

Definition 23.1 (Bayesian incentive compatibility (BIC)) *(f, p) is BIC if for all $i \in N$ and all $s_i, t_i \in T_i$, $t_i \alpha_i(t_i; t_i) - \pi_i(t_i; t_i) \geqslant t_i \alpha_i(s_i; t_i) - \pi_i(s_i; t_i)$. f is* Bayesian implementable *if some p makes (f, p) BIC.*

Definition 23.2 (Interim individual rationality (IIR)) *(f, p) is IIR if for all i and all $t_i \in T_i$, $t_i \alpha_i(t_i; t_i) - \pi_i(t_i; t_i) \geqslant 0$.*

From now on, priors are **independent**: $G(s_1, \ldots, s_n) = \prod_i G_i(s_i)$ and $G_{-i}(s_{-i} \mid t_i) = \prod_{j \neq i} G_j(s_j)$. We write $\alpha_i(t_i)$ and $\pi_i(t_i)$ for the truthful quantities.

Definition 23.3 (Non-decreasing in expectation (NDE)) *f is NDE if for all i and all $s_i < t_i$, $\alpha_i(s_i) \leqslant \alpha_i(t_i)$.*

Only the *expected* allocation must be monotone. A pointwise non-decreasing (ND) f is NDE, but an NDE f need not be ND.

Theorem 23.4 (Myerson 1981) *With independent priors, (f, π) is BIC iff (1) f is NDE, and (2) for all i and t_i, $\pi_i(t_i) = \pi_i(0) + t_i\alpha_i(t_i) - \int_0^{t_i} \alpha_i(x)\,dx$.*

Lemma 23.5 *A mechanism is BIC and IIR iff (1) f is NDE, (2) π_i follows Myerson's payment formula, and (3) $\pi_i(0) \leqslant 0$ for all i.*

Proof: Conditions (1) and (2) are exactly BIC. So we show IIR $\iff$ $\pi_i(0) \leqslant 0$. ($\Rightarrow$) IIR at $t_i = 0$ gives $0 \cdot \alpha_i(0) - \pi_i(0) \geqslant 0$. ($\Leftarrow$) Substituting the payment formula, the utility is $-\pi_i(0) + \int_0^{t_i} \alpha_i(s)\,ds$. The integral is non-negative, and so is $-\pi_i(0)$. ■

23.2 One Buyer: Revenue Becomes Virtual Value

Example 23.6 (World Cup broadcast rights) *A cricket board owns the TV rights to the World Cup in one country. The rights are indivisible: exactly one channel airs the matches. Say only one sports channel there can afford them, so there is a single buyer. The channel's value t depends on its advertising deals, and only the channel knows it. The board knows only the prior G. How should it sell the rights to earn the most, on average?*

The single buyer has type $t \in T = [0, \beta]$ with CDF G and density g. A mechanism $M = (f, p)$ has $f : [0, \beta] \to [0, 1]$ (probability of getting the rights) and $p : [0, \beta] \to \mathbb{R}$. With one agent, BIC and DSIC coincide, and so do IR and IIR:

$$\begin{aligned} &\text{IC:} \quad tf(t) - p(t) \geqslant tf(s) - p(s), \quad \forall s, t \in T, \\ &\text{IR:} \quad tf(t) - p(t) \geqslant 0, \quad \forall t \in T. \end{aligned}$$

The expected revenue is $\Pi^M := \int_0^{\beta} p(t)g(t)\,dt$.

Definition 23.7 (Optimal mechanism) *M^* is optimal if it is IC and IR and $\Pi^{M^*} \geqslant \Pi^M$ for every IC and IR mechanism M.*

By the characterization, f is non-decreasing and $p(t) = p(0) + tf(t) - \int_0^t f(x)\,dx$, with $p(0) \leqslant 0$ for IR.

Why $p(0) = 0$. Here is a specific example. Suppose an IC mechanism pays the zero type, say $p(0) = -5$ (for instance, post a price and also hand 5 to every buyer). IR still holds. Now add 5 to $p(t)$ for every t. A constant shift changes no IC comparison. The utility becomes $\int_0^t f(x)\,dx \geqslant 0$, so IR holds. Revenue rises by 5. Hence an optimal mechanism has $p(0) = 0$, and

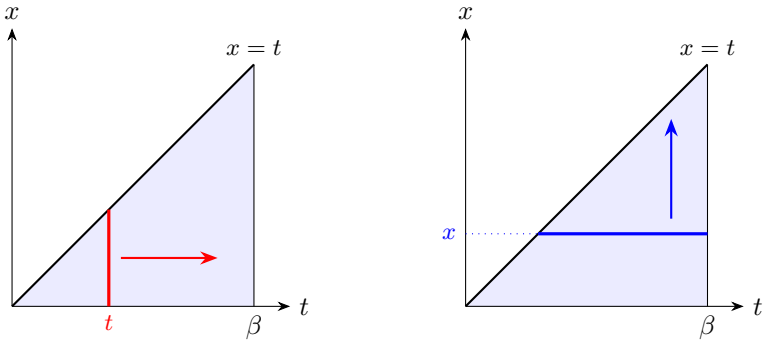
$$p(t) = tf(t) - \int_0^t f(x)\,dx.$$

The payment is fixed once f is fixed. So we optimize over f alone.

23.2.1 The revenue identity

$$\begin{aligned}\Pi^f &= \int_0^\beta \left(tf(t) - \int_0^t f(x)\,dx \right) g(t)\,dt = \int_0^\beta tf(t)g(t)\,dt - \int_0^\beta \int_0^t f(x)\,g(t)\,dx\,dt \\ &= \int_0^\beta tf(t)g(t)\,dt - \int_0^\beta \left(\int_x^\beta g(t)\,dt \right) f(x)\,dx \qquad \text{[switching the order of integration]} \\ &= \int_0^\beta tf(t)g(t)\,dt - \int_0^\beta \big(1 - G(x)\big) f(x)\,dx = \int_0^\beta \left(t - \frac{1-G(t)}{g(t)} \right) f(t)\,g(t)\,dt.\end{aligned}$$

The switch is over the triangle $0 \leqslant x \leqslant t \leqslant \beta$ (Figure 23.1). Slicing it vertically fixes t and runs x over $[0, t]$. Slicing it horizontally fixes x and runs t over $[x, \beta]$, which gives $\int_x^\beta g(t)\,dt = 1 - G(x)$.



fix t, $x \in [0, t]$; slide t from 0 to β fix x, $t \in [x, \beta]$; slide x from 0 to β

Figure 23.1: One slice in each order. Sweeping the slice covers the same triangle.

Lemma 23.8 *For any implementable f, $\Pi^f = \int_0^\beta w(t) f(t) g(t)\,dt$, where*

$$w(t) = t - \frac{1-G(t)}{g(t)}$$

*is the **virtual valuation** of the agent.*

The seller cannot collect the full value t. Part of it must be left to the buyer to keep her truthful (an *information rent*). The virtual value $w(t)$ is what is left for the seller.

23.3 MHR and the Posted Price

The problem is now

$$\textbf{OPT1:} \qquad \max_{f \text{ non-decreasing}} \int_0^\beta \left(t - \frac{1-G(t)}{g(t)} \right) g(t) f(t)\,dt.$$

Definition 23.9 (Monotone hazard rate (MHR)) *G satisfies MHR if $\frac{g(x)}{1-G(x)}$ is non-decreasing in x.*

Uniform and exponential priors satisfy MHR. For $U[0,\beta]$, $\frac{g}{1-G} = \frac{1}{\beta - x}$, which increases. For the exponential with mean μ, $\frac{g}{1-G} = \frac{1}{\mu}$, a constant, so $\frac{1-G(x)}{g(x)} = \mu$.

Under MHR, $\frac{1-G(x)}{g(x)}$ is non-increasing while x increases. So the equation $x = \frac{1-G(x)}{g(x)}$ has a solution x^*, and

$$w(x) < 0 \ \ \forall x < x^*, \qquad w(x^*) = 0, \qquad w(x) > 0 \ \ \forall x > x^*.$$

The pointwise maximizer of the integrand sets $f(t) = 0$ for $t < x^*$, $f(t) = 1$ for $t > x^*$, and any $f(x^*) \in [0,1]$. This f is non-decreasing, so it solves OPT1.

Theorem 23.10 *Suppose G satisfies MHR. A mechanism (f,p) is optimal iff (1) $f(t) = 1$ if $t \geqslant x^*$ and 0 otherwise, where $x^* = \frac{1-G(x^*)}{g(x^*)}$, and (2) $p(t) = x^*$ if $t \geqslant x^*$ and 0 otherwise.*

In words, the optimal mechanism is a **take-it-or-leave-it posted price** x^*.

Example 23.11 (World Cup broadcast rights, continued) *Let $t \sim U[0,1]$ (in thousand crore rupees). Then $G(t) = t$, $g(t) = 1$, and $w(t) = t - (1-t) = 2t - 1$. So $x^* = \frac{1}{2}$. The board posts the price $\frac{1}{2}$, i.e., 500 crore rupees. The channel buys with probability $\frac{1}{2}$, and the expected revenue is $\frac{1}{4}$, i.e., 250 crore rupees.*

A shortcut for one buyer. *With a single channel, we could simply have guessed a posted price q. The channel buys iff $t \geqslant q$, so the revenue is $q(1-q)$. A low price sells often but earns little; a high price earns a lot but rarely sells. The peak is at $q = \frac{1}{2}$, the same answer as x^*. But the shortcut searches only over posted prices, and only works for one buyer. With two or more channels, the board could run an auction, set reserves, favour one channel, or even randomize, and there is nothing simple to guess. Myerson's route,* revenue equals expected virtual surplus, *works for any number of buyers. It also proves (Theorem 23.10) that for one buyer no other IC and IR mechanism, however clever, beats the posted price.*

23.4 Many Buyers: Maximize Virtual Surplus

Example 23.12 (Two channels want the rights) *Back to the cricket board of Example 23.6. Now two channels can afford the World Cup TV rights: a regional channel (1) and a national network (2). The rights are still indivisible, so exactly one of them airs the matches. Each channel knows its own value, and the board does not. But the board knows who they are. The national network reaches more viewers, so its value range is wider. The priors are independent but different.*

A mechanism is optimal if it is BIC, IIR, and maximizes expected revenue. By Lemma 23.5, this means f is NDE, π_i follows Myerson's formula, and $\pi_i(0) = 0$. Agent i's expected payment is $\int_0^{b_i} \pi_i(t_i) g_i(t_i)\, dt_i$. Exactly as in the single agent case, this equals

$$\int_0^{b_i} w_i(t_i)\, \alpha_i(t_i)\, g_i(t_i)\, dt_i, \qquad w_i(t_i) := t_i - \frac{1 - G_i(t_i)}{g_i(t_i)}.$$

Now substitute $\alpha_i(t_i) = \int_{T_{-i}} f_i(t_i, t_{-i}) g_{-i}(t_{-i})\, dt_{-i}$. By **independence**, $g_i(t_i) g_{-i}(t_{-i}) = g(t)$, so the expected payment is $\int_T w_i(t_i) f_i(t) g(t)\, dt$. Summing over agents,

$$\Pi^f = \int_T \left(\sum_{i \in N} w_i(t_i) f_i(t) \right) g(t)\, dt,$$

the expected total **virtual surplus**. The problem becomes

$$\textbf{OPT2:} \qquad \max_{f \text{ NDE}} \int_T \left(\sum_{i \in N} w_i(t_i) f_i(t) \right) g(t)\, dt.$$

Drop the NDE constraint and maximize pointwise:

$$\begin{aligned} f_i(t) &= \begin{cases} 1 & \text{if } w_i(t_i) \geqslant w_j(t_j)\ \forall j \text{ (ties broken arbitrarily)}, \\ 0 & \text{otherwise}, \end{cases} && \textbf{(Sold)} \\ f_i(t) &= 0 \quad \forall i \in N, \quad \text{if } w_i(t_i) < 0\ \forall i \in N && \textbf{(Unsold)}. \end{aligned} \tag{23.1}$$

This rule need not be NDE (see Example 23.14 below, and Myerson's 1981 paper). It fails when the following condition fails.

Definition 23.13 (Regularity) *w_i is **regular** if for all $s_i < t_i$ in T_i, $w_i(s_i) \leqslant w_i(t_i)$.*

Example 23.14 (When regularity fails, NDE breaks) *Take two agents. Agent j has $t_j \sim U[0,1]$, so $w_j(t_j) = 2t_j - 1$, which is regular. Agent i has a virtual valuation that rises, dips, and rises again: the piecewise linear w_i through $(0,-0.6)$, $(1,0.6)$, $(2,0.2)$, $(3,1)$ (Figure 23.2, left).*

Fix $t_j = 0.7$, so $w_j = 0.4$. At $t_i = 1$, $w_i = 0.6 > 0.4$ and i wins. At $t_i' = 2$, $w_i = 0.2 < 0.4$ and i loses. So a higher type loses where a lower type won, and f_i is not non-decreasing.

Averaging over t_j does not help. For $w_i(t_i) \geqslant 0$,

$$\alpha_i(t_i) = \Pr\left[w_j(t_j) \leqslant w_i(t_i)\right] = \Pr\left[t_j \leqslant \tfrac{1+w_i(t_i)}{2}\right] = \frac{1 + w_i(t_i)}{2},$$

and $\alpha_i(t_i) = 0$ when $w_i(t_i) < 0$. So α_i copies the dip in w_i (Figure 23.2, right): $\alpha_i(1) = 0.8 > \alpha_i(2) = 0.6$, and NDE fails.

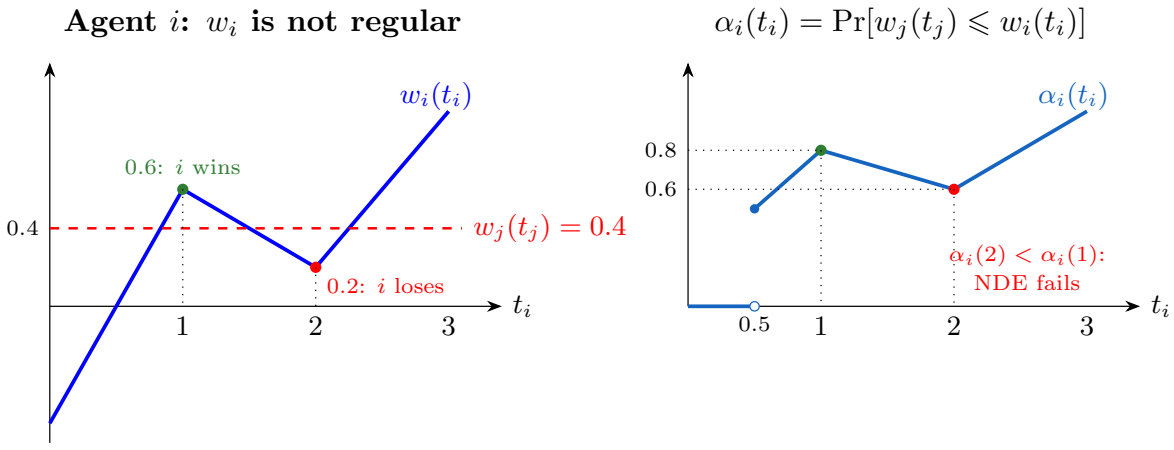


Figure 23.2: A non-regular w_i (left) makes the interim allocation α_i non-monotone (right).

Regularity is weaker than MHR. Under MHR, $\frac{1-G_i}{g_i}$ is non-increasing, and t_i increases. So w_i, their difference, is non-decreasing. The converse need not hold.

Lemma 23.15 *If every w_i is regular, the optimal allocation equals the unconstrained solution (23.1).*

Proof: Take f from (23.1). If $s_i < t_i$, regularity gives $w_i(s_i) \leqslant w_i(t_i)$. So if i wins at (s_i, t_{-i}), she also wins at (t_i, t_{-i}). Hence $f_i(t_i, t_{-i}) \geqslant f_i(s_i, t_{-i})$ for all t_{-i}. So f_i is non-decreasing, hence NDE, and the unconstrained optimum is feasible for OPT2. ∎

The solution is deterministic and pointwise non-decreasing, which is stronger than we needed.

23.5 Myerson's Optimal Auction

Theorem 23.16 (Myerson 1981) *Suppose every w_i is regular. For every profile t:*

- *If $w_i(t_i) < 0$ for all i, the item is not sold.*
- *Otherwise, $f_i(t) = 1$ if $w_i(t_i) \geqslant w_j(t_j)$ for all j (ties broken arbitrarily), and 0 otherwise.*
- *Payments are $p_i(t) = 0$ if $f_i(t) = 0$, and $p_i(t) = \max\{w_i^{-1}(0), K_i^*(t_{-i})\}$ if $f_i(t) = 1$, where $w_i^{-1}(0)$ is the type with $w_i = 0$ and $K_i^*(t_{-i}) := \inf\{t_i : f_i(t_i, t_{-i}) = 1\}$.*

Then (f, p) is an optimal mechanism.

$K_i^*(t_{-i})$ is the smallest type at which i starts winning. The winner clears two bars, her own *reserve* $w_i^{-1}(0)$ and the others' virtual values. She pays the smallest bid that clears both, which does not depend on her own bid.

Example 23.17 (Regional vs. national) *In Example 23.12, let $T_1 = [0, 12]$ (regional) and $T_2 = [0, 18]$ (national), in hundred crore rupees, with independent uniform priors. Then $g_1 = \frac{1}{12}$, $G_1(t_1) = \frac{t_1}{12}$, $g_2 = \frac{1}{18}$, $G_2(t_2) = \frac{t_2}{18}$, and*

$$w_1(t_1) = t_1 - \frac{1 - \frac{t_1}{12}}{\frac{1}{12}} = 2t_1 - 12, \qquad w_2(t_2) = 2t_2 - 18.$$

The reserves are 6 for the regional channel and 9 for the national network. Also $w_1 \geqslant w_2 \iff t_1 \geqslant t_2 - 3$. Hence

$$p_1 = K_1^*(t_2) = \max\{6, t_2 - 3\}, \qquad p_2 = K_2^*(t_1) = \max\{9, t_1 + 3\}.$$

t_1	t_2	*Virtual values*	*Action*	p_1	p_2
4	*8*	*both $w < 0$*	*unsold*	*0*	*0*
2	*12*	$w_2 > 0$	*sold to 2*	*0*	$\max(9, 2+3) = 9$
6	*6*	$w_1 = 0$, $w_2 < 0$	*sold to 1*	$\max(6, 6-3) = 6$	*0*
9	*9*	$w_1 = 6$, $w_2 = 0$	*sold to 1*	$\max(6, 9-3) = 6$	*0*
8	*15*	$w_1 = 4$, $w_2 = 12$	*sold to 2*	*0*	$\max(9, 8+3) = 11$

At $(9, 9)$ the values are equal, yet the regional channel wins. The weaker bidder is favoured, which pushes the national network to pay more. At $(4, 8)$ both value the rights, but nobody airs the matches. Figure 23.3 shows the regions. In the hatched region, the regional channel wins although the national network values the rights more.

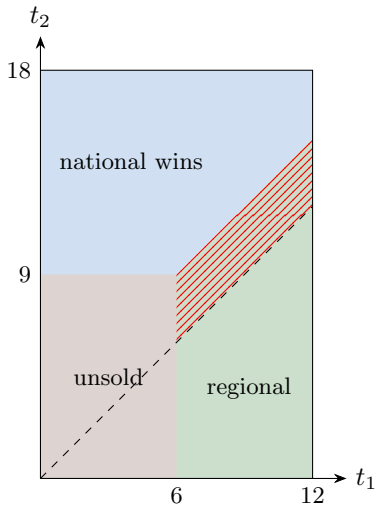


Figure 23.3: Optimal allocation for Example 23.17. The dashed line is $t_1 = t_2$.

23.5.1 Beyond regularity: ironing

Theorem 23.16 assumes every w_i is regular. Many priors are not, for instance a mix of two kinds of buyers. Then w_i dips somewhere, and "highest w_i wins" can violate NDE: a higher report may lower w_i and lose. Myerson's fix is to **iron** the virtual valuation. Over each interval $[a, b]$ where w_i dips, replace w_i by a constant, namely its g_i-weighted average over that interval. The result $\bar{w}_i$, the *ironed virtual valuation*, is non-decreasing (Figure 23.4). Formally, the ironing is done in quantile space, by taking the convex hull of the integral of w_i. The item goes to the highest non-negative $\bar{w}_i$, with ties inside a flat interval broken at random. The resulting mechanism is optimal for any independent priors, and payments again follow Myerson's formula. See [Mye81], Section 6, and [Har], Section 3.3.5.

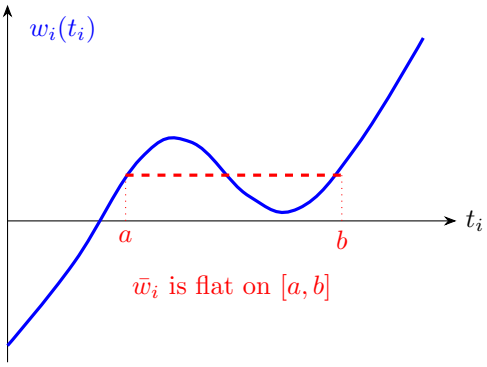


Figure 23.4: An irregular virtual valuation (solid) and its ironed version (dashed over $[a, b]$).

23.6 Summary

The revenue of any BIC and IIR mechanism equals its expected virtual surplus (Lemma 23.8). For one buyer with an MHR prior, the optimal mechanism is a posted price x^* with $w(x^*) = 0$ (Theorem 23.10). With

many buyers and regular priors, the item goes to the highest non-negative virtual value, and the winner pays $\max\{w_i^{-1}(0), K_i^*(t_{-i})\}$ (Theorem 23.16). For irregular priors, the same rule works with ironed virtual valuations. The optimal auction is not efficient. Items may stay unsold, and the bidder with the highest value may lose. Next lecture, we see more examples, and then ask what happens when the designer insists on efficiency instead.

References

[Har] J. D. Hartline, *Mechanism Design and Approximation*, Chapter 3 (Optimal Mechanisms), book draft available online.

[Mye81] R. B. Myerson, "Optimal auction design," *Mathematics of Operations Research*, 6(1), 1981, pp. 58–73.