CS 405/6001: Game Theory and Algorithmic Mechanism Design **Jul–Nov 2026**

Lecture 24: Groves and dAGVA Mechanisms

Lecturer: Swaprava Nath *Scribe(s): Claude Scribe*

Disclaimer: *These notes aggregate content from several texts and have not been subjected to the usual scrutiny deserved by formal publications. If you find errors, please bring to the notice of the Instructor.*

24.1 Recap and Plan

Last lecture the designer wanted revenue. With a single buyer and an MHR prior, the optimal mechanism posts a price x^* with $x^* = \frac{1-G(x^*)}{g(x^*)}$. With many buyers and regular virtual valuations $w_i(t_i) = t_i - \frac{1-G_i(t_i)}{g_i(t_i)}$, the item goes to the highest non-negative virtual value, and the winner pays $\max\{w_i^{-1}(0), K_i^*(t_{-i})\}$, where $K_i^*(t_{-i})$ is the type at which i starts winning.

Today runs in four chunks:

1. Two more examples of Myerson's auction: reserve prices and inefficiency (Section 24.2).
2. Every efficient and DSIC mechanism is Groves (Section 24.3).
3. No Groves mechanism is budget balanced (Section 24.4).
4. Relaxing to BIC: dAGVA, Myerson–Satterthwaite, and the space of mechanisms (Sections 24.5–24.6).

24.2 Reserve Prices and (In)efficiency

Example 24.1 (Channels of equal size: symmetric bidders) *The cricket board sells the TV rights to a domestic T20 league. Several national channels of similar size bid, so their values come from the same distribution: $g_i = g$ and $T_i = T$ for all i. So $w_i = w$ for all i. With regularity, $w(t_i) > w(t_j)$ iff $t_i > t_j$. Hence the rights go to the highest bidder. It is not sold if $t_i < w^{-1}(0)$ for every i. The winner pays*

$$p_i = \max\left\{w^{-1}(0), \max_{j \neq i} t_j\right\}.$$

*This is a **second price auction with a reserve price**. It is efficient whenever the rights are sold.*

For instance, let the values be i.i.d. $U[0, 100]$ (in crore rupees). Then $w(t) = 2t - 100$ and the reserve is 50. With bids $(80, 60)$, the first channel wins and pays $\max(50, 60) = 60$. With bids $(45, 30)$, nobody reaches 50, so the rights are unsold. With bids $(65, 20)$, the first channel pays the reserve 50 instead of 20. The reserve sometimes costs a sale, but on average it raises revenue.

Example 24.2 (TV vs. streaming: efficiency vs. optimality) *The board sells the rights either to a TV channel (1) with $T_1 = [0, 10]$ or to a streaming app (2) with $T_2 = [0, 6]$, with independent uniform priors. So $w_1(t_1) = 2t_1 - 10$ and $w_2(t_2) = 2t_2 - 6$. The app wins iff $t_2 \geqslant 3$ and $t_2 \geqslant t_1 - 2$.*

The allocation is inefficient when the rights stay unsold. It is also inefficient in the region where the TV channel has the higher value but the rights go to the app (Figure 24.1). For example, at $(t_1, t_2) = (7, 6)$,

$w_1 = 4$ and $w_2 = 6$, so the app wins and pays $K_2^*(t_1) = \max\{3, t_1 - 2\} = 5$, although the TV channel values the rights more.

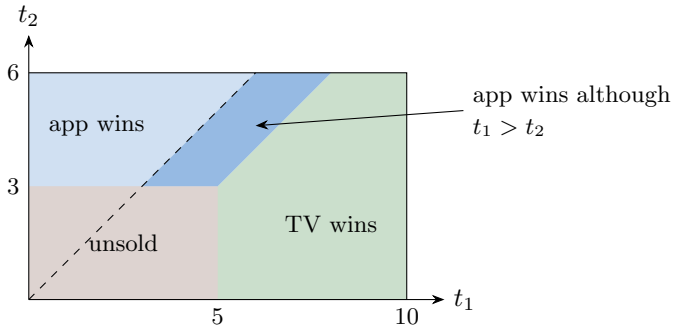


Figure 24.1: Optimal allocation in Example 24.2. The dashed line is $t_1 = t_2$.

So the revenue-optimal mechanism trades efficiency for revenue. For the rest of the lecture, we insist on efficiency and ask what truthfulness then forces.

24.3 Efficient and DSIC Means Groves

Example 24.3 (Kick-off time for the World Cup final) *The cricket board picks one kick-off time for the final: afternoon (a) or evening (b). Everyone lives with the choice. Three parties care, and each has a private value (in crore rupees) for each time. The TV channel earns more from prime-time ads, so it loves evening. The stadium saves on lighting, so it prefers afternoon. An overseas streaming partner prefers afternoon because of its time zone. Say the values are*

	Afternoon a	*Evening b*
TV channel	*1*	*5*
Stadium	*4*	*1*
Streamer	*3*	*1*
Total	*8*	*7*

Efficiency picks afternoon, 8 *against* 7*. Without money, though, the channel can report* 10 *for evening and flip the decision. So the board needs payments that make such lies useless (DSIC).*

An allocation rule is **efficient** if it maximizes total welfare:

$$f^{\text{eff}}(t) \in \arg\max_{a \in A} \sum_{i \in N} t_i(a).$$

Theorem 24.4 (Green and Laffont 1979; Holmström 1979) *If the type space is sufficiently rich, every efficient and DSIC mechanism is a Groves mechanism.*

"Sufficiently rich" means that for every agent i and alternative x, the coordinate $t_i(x)$ can vary freely over an open interval. This is a uniqueness result. Truthfulness and efficiency together leave only Groves payments.

Proof: Take two alternatives $A = \{a, b\}$, and suppose $\sum_i t_i(a) \geqslant \sum_i t_i(b)$, so a is chosen. Fix t_{-i} and fix $t_i(b)$. By efficiency, raising $t_i(a)$ never makes a lose. So there is a threshold $t_i^*(a)$ such that a is chosen for all $t_i(a) \geqslant t_i^*(a)$, and b for all $t_i(a) < t_i^*(a)$. (With more alternatives, compare a with each $c \neq a$ and take the largest pairwise threshold.)

In Example 24.3, take the stadium as agent i. Its evening value is $t_i(b) = 1$, and the channel and the streamer together have $\sum_{j \neq i} t_j(a) = 4$ and $\sum_{j \neq i} t_j(b) = 6$. Then afternoon wins iff $t_i(a) + 4 \geqslant 1 + 6$, so $t_i^*(a) = 3$. The stadium's true afternoon value is $4 \geqslant 3$, so afternoon is chosen.

Let p_{ia} and p_{ib} be i's payments at a and b. (Given t_{-i}, the payment can depend on i's report only through the outcome.) DSIC at the true type $t_i(a) = t_i^*(a) + \epsilon$, $\epsilon > 0$, gives $t_i^*(a) + \epsilon - p_{ia} \geqslant t_i(b) - p_{ib}$. DSIC at $t_i'(a) = t_i^*(a) - \delta$, $\delta > 0$, gives $t_i(b) - p_{ib} \geqslant t_i^*(a) - \delta - p_{ia}$. Since ϵ and δ are arbitrary,

$$t_i^*(a) - p_{ia} = t_i(b) - p_{ib}. \tag{24.1}$$

At the threshold, efficiency is indifferent:

$$t_i^*(a) + \sum_{j \neq i} t_j(a) = t_i(b) + \sum_{j \neq i} t_j(b). \tag{24.2}$$

Subtracting (24.2) from (24.1),

$$p_{ia} - p_{ib} = \sum_{j \neq i} t_j(b) - \sum_{j \neq i} t_j(a).$$

So $p_{ix} = h_i(t_{-i}) - \sum_{j \neq i} t_j(x)$ for some function h_i that does not depend on i's report. In the kick-off example, the stadium's $p_{ia} - p_{ib} = 6 - 4 = 2$. This is exactly the welfare the channel and the streamer lose when afternoon is picked over evening. ■

Definition 24.5 (Groves mechanism) *(f, p) is a Groves mechanism if $f(t) \in \arg\max_{a \in A} \sum_i t_i(a)$ and*

$$p_i(t) = h_i(t_{-i}) - \sum_{j \neq i} t_j(f(t)).$$

Under a Groves payment, i's utility is the total welfare minus $h_i(t_{-i})$. So i wants the efficient outcome too. If h_i depended on i's own report, this alignment would break.

Back to the lying channel in Example 24.3. Take h_i to be the others' best welfare, $h_i(t_{-i}) = \max_x \sum_{j \neq i} t_j(x)$ (the Clarke choice). For the channel, the stadium and the streamer have welfare 7 at afternoon and 2 at evening, so $h_i = 7$. If the channel truthfully reports, afternoon wins and it pays $7 - 7 = 0$, for a utility of 1. If it reports 10 for evening, evening wins and it pays $7 - 2 = 5$, for a true utility of $5 - 5 = 0$. The lie does not pay.

24.4 Groves Cannot Balance the Budget

Example 24.6 (The DAO vote on a public zk-prover) *A DAO decides whether to fund a public zk-prover (outcome 1) or not (outcome 0). Every member can use it. Each member's value for funding is private, and at outcome 0 every value is zero. The DAO wants the efficient decision, truthful votes, and no money left over or missing. That last property is budget balance.*

Theorem 24.7 (Green and Laffont 1979) *If the type space is sufficiently rich, no Groves mechanism is budget balanced. That is, there are no payments p_i^G with $\sum_{i \in N} p_i^G(t) = 0$ for all $t \in T$.*

Corollary 24.8 *If the valuation space is sufficiently rich, no efficient mechanism can be both DSIC and budget balanced.*

Proof: Use the public project of Example 24.6 with two agents, and let w_1, w_2 be their values for outcome 1. Suppose some h_1, h_2 make the Groves payments budget balanced. Pick two types w_1^+, w_1^- for agent 1 and one type w_2 for agent 2 such that

$$w_1^+ + w_2 > 0 \ \text{(built)}, \qquad w_1^- + w_2 < 0 \ \text{(not built)}.$$

Budget balance at (w_1^+, w_2) gives $h_1(w_2) - w_2 + h_2(w_1^+) - w_1^+ = 0$. At (w_1^-, w_2) it gives $h_1(w_2) + h_2(w_1^-) = 0$. Eliminating $h_1(w_2)$,

$$w_2 = h_2(w_1^+) - h_2(w_1^-) - w_1^+.$$

The right side does not depend on w_2. Increase w_2 by a small $\delta > 0$ so that both inequalities still hold. Both decisions stay the same, and the same argument gives $w_2 + \delta$ equal to the same right side. This is a contradiction. ∎

24.5 Weakening DSIC: the dAGVA Mechanism

To recover budget balance, we relax DSIC to BIC.

Example 24.9 (Three studios, one render farm) *Three game studios share a render farm. The schedule should maximize total value, and money should move only between the studios.*

Definition 24.10 (dAGVA mechanism; d'Aspremont and Gérard-Varet 1979, Arrow 1979) *The allocation is efficient:* $a^*(t) \in \arg\max_{a \in A} \sum_i t_i(a)$. *For each* i, *define*

$$\delta_i(t_i) := \mathbb{E}_{t_{-i}|t_i}\Big[\sum_{j \neq i} t_j(a^*(t))\Big],$$

and charge

$$p_i^{dAGVA}(t) = \frac{1}{n-1}\sum_{j \neq i} \delta_j(t_j) - \delta_i(t_i).$$

Theorem 24.11 *The dAGVA mechanism is efficient, BIC, and budget balanced.*

Proof: *BIC.* The expected utility of truthful i is

$$\begin{aligned}
\mathbb{E}_{t_{-i}|t_i}\big[t_i(a^*(t)) - p_i^{\mathrm{dAGVA}}(t)\big] &= \mathbb{E}_{t_{-i}|t_i}\sum_{j \in N} t_j(a^*(t)) - \mathbb{E}_{t_{-i}|t_i}\Big[\frac{1}{n-1}\sum_{j \neq i}\delta_j(t_j)\Big] \\
&\geqslant \mathbb{E}_{t_{-i}|t_i}\sum_{j \in N} t_j(a^*(t_i', t_{-i})) - \mathbb{E}_{t_{-i}|t_i}\Big[\frac{1}{n-1}\sum_{j \neq i}\delta_j(t_j)\Big] \\
&= \mathbb{E}_{t_{-i}|t_i}\big[t_i(a^*(t_i', t_{-i})) - p_i^{\mathrm{dAGVA}}(t_i', t_{-i})\big].
\end{aligned}$$

The first equality holds because $-\delta_i(t_i)$ adds back the others' expected welfare. The inequality holds because a^* maximizes total welfare pointwise. The second term does not depend on i's report.

Budget balance.

$$\sum_{i\in N} p_i^{\mathrm{dAGVA}}(t) = \frac{1}{n-1}\sum_{i\in N}\sum_{j\neq i}\delta_j(t_j) - \sum_{i\in N}\delta_i(t_i) = \frac{n-1}{n-1}\sum_{j\in N}\delta_j(t_j) - \sum_{i\in N}\delta_i(t_i) = 0,$$

since each δ_j appears $n-1$ times in the double sum. ■

In Example 24.9, say $n=3$ and $\delta=(4,2,6)$. Then $p_1 = \frac{2+6}{2} - 4 = 0$, $p_2 = \frac{4+6}{2} - 2 = 3$, and $p_3 = \frac{4+2}{2} - 6 = -3$. The payments add up to zero. Studio 2 pays, studio 3 receives, and the farm keeps nothing.

Remark 24.12 *dAGVA is efficient, BIC and budget balanced, but it is* not *IIR. Some types may be worse off by participating.*

Example 24.13 (The used-GPU trade) *One student sells a used GPU to another. The seller's cost c and the buyer's value v are private. Efficiency requires trade iff $v > c$.*

Theorem 24.14 (Myerson and Satterthwaite 1983) *In bilateral trade between a seller and a buyer, no mechanism is simultaneously BIC, efficient, IIR, and budget balanced.*

So if the resale platform insists on BIC, efficiency and IIR, the payments cannot always sum to zero. Someone outside has to cover the gap. Relaxing DSIC to BIC does not remove the tension between efficiency, participation and budget balance.

24.6 The Space of Mechanisms

Imposing more properties shrinks the set of feasible mechanisms. Relaxing them, for example by introducing a prior, enlarges it.

Deterministic DSIC mechanisms. With quasi-linear preferences, DSIC mechanisms include dictatorial mechanisms and affine maximizers. Adding efficiency collapses the class to exactly the Groves mechanisms (Theorem 24.4). This is one of the few two-way results in mechanism design. Adding budget balance on top empties the class (Theorem 24.7). See Figure 24.2.

BIC mechanisms. Relaxing DSIC to BIC enlarges the space, because incentive constraints hold only in expectation. Every DSIC mechanism is BIC, but not conversely. Here dAGVA is efficient, BIC and budget balanced. The Myerson–Satterthwaite theorem marks the frontier where IIR is added as well. Myerson's optimal auctions also live here, and they trade efficiency for revenue. See Figures 24.3 and 24.4.

24.7 Summary

- With symmetric regular bidders, the optimal auction is a second price auction with a reserve (Example 24.1). With asymmetric bidders it can be inefficient (Example 24.2).
- Every efficient and DSIC mechanism is a Groves mechanism (Theorem 24.4).
- No Groves mechanism is budget balanced (Theorem 24.7).

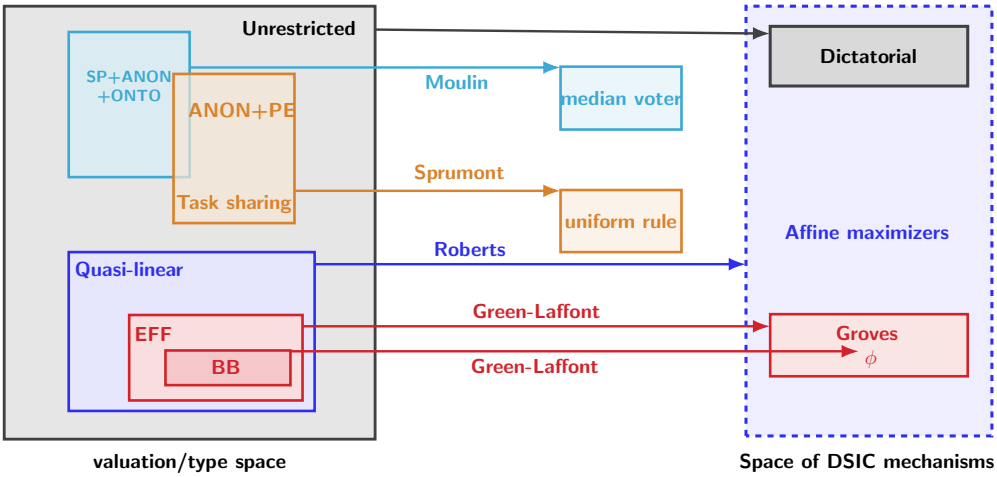


Figure 24.2: Space of DSIC mechanisms.

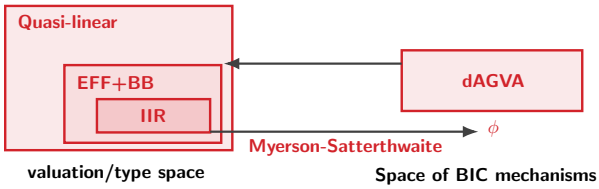


Figure 24.3: Space of BIC mechanisms.

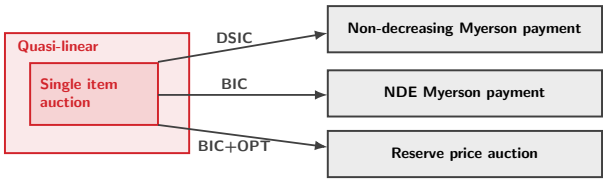


Figure 24.4: DSIC, BIC, and optimal mechanisms.

- Relaxing DSIC to BIC allows efficiency with budget balance (dAGVA, Theorem 24.11), but not IIR.
- In bilateral trade, BIC, efficiency, IIR and budget balance cannot all hold (Theorem 24.14).

Mechanism design is a game of trade-offs among truthfulness, efficiency, participation, and budget.

References

[Arr79] K. J. Arrow, "The property rights doctrine and demand revelation under incomplete information," in *Economics and Human Welfare*, Academic Press, 1979.

[dAGV79] C. d'Aspremont and L.-A. Gérard-Varet, "Incentives and incomplete information," *Journal of Public Economics*, 11(1), 1979, pp. 25–45.

[GL79] J. Green and J.-J. Laffont, *Incentives in Public Decision-Making*, North-Holland, 1979.

[Hol79] B. Holmström, "Groves' scheme on restricted domains," *Econometrica*, 47(5), 1979, pp. 1137–1144.

[MS83] R. B. Myerson and M. A. Satterthwaite, "Efficient mechanisms for bilateral trading," *Journal of Economic Theory*, 29(2), 1983, pp. 265–281.