भारतीय प्रौद्योगिकी संस्थान मुंबई
Indian Institute of Technology Bombay

CS 6001: Game Theory and Algorithmic Mechanism Design

Week 1

Swaprava Nath

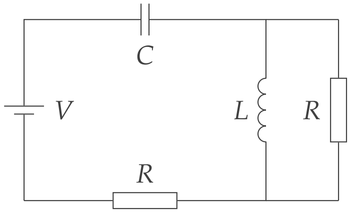
Slide preparation acknowledgments: Ramsundar Anandanarayanan and Harshvardhan Agarwal

ज्ञानम् परमम् ध्येयम्
Knowledge is the supreme goal

Contents

Typical Engineering Courses

- Circuit **analysis**



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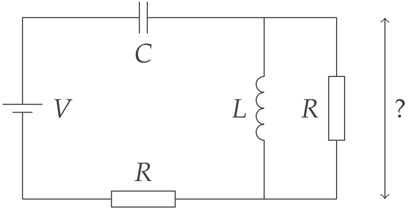

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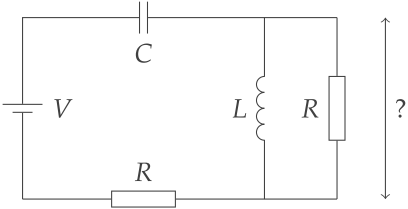

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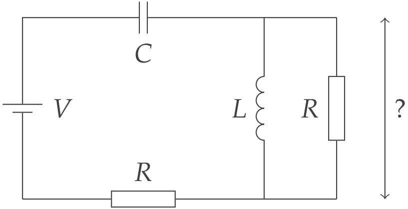

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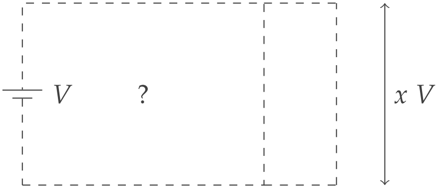

synthesis

Similarly ...

Given Game $\longrightarrow$

Similarly ...

Given Game ⟶ Outcomes ?

Similarly ...

$$\boxed{\text{Given Game}} \longrightarrow \text{Outcomes ?}$$

Game Theory

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Given Game $\longrightarrow$ Outcomes ? $\longrightarrow$ Outcomes

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Mechanism Design

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Mechanism Design

- Social **analysis** and **synthesis**

Contents

Game: Neighboring Kingdom's Dilemma

		Rashtrakuta: Agri	Rashtrakuta: War
Pala	Agri	5, 5	0, 6
Pala	War	6, 0	1, 1

Game: Neighboring Kingdom's Dilemma

Pala \ Rashtrakuta	Agri	War
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Question

What is a reasonable outcome of this game?

Game

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  - Not in multi-state games
- Games can be of many *kinds* and *representations*:
  Normal form, **Extensive form**, **Static**, **Dynamic**, **Repeated**, **Stochastic**, ...

Game Theory

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- **Goal of game theory:** **predict** the outcomes of a game (refer to the dilemma game)

Assumptions of Game Theory

This course is an axiomatic analysis of multi-agent behavior – and the axioms are as follows

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[1]This person is correct beyond any question. Whatever he says must be true.

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Question

How does common knowledge percolate?

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Percolation of Common Knowledge

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Assumption in Game Theory

The fact that all players are rational and intelligent is a common knowledge

Contents

Game of Chess

Description

- Two-player game: White (**W**) and Black (**B**) – 16 pieces each

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  1. **Win** for **W**: if W captures B king
  2. **Win** for **B**: if B captures W king
  3. **Draw**: everything else, e.g., if nobody has legal moves but kings are not in check, both players agree to a draw, board position is such that nobody can win, ...

Natural Questions from Game Theorist's perspective

Question

Does **W** have a winning strategy?
i.e., a plan of moves such that it wins **irrespective** of the moves of **B**?

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Or do either have at least a draw guaranteeing strategy?

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Or do either have at least a draw guaranteeing strategy?

- Neither may be possible – not synonymous with the end of the game

What is a strategy?

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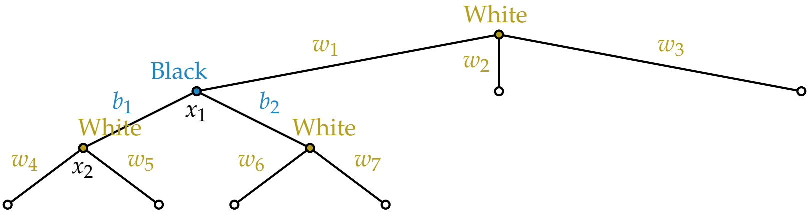
  - $x_k \to x_{k+1}$

    k even – created by a single action of W

    k odd – created by a single action of B

What is a strategy? (contd.)

Board positions may repeat in this tree, but a vertex is unique – **game situation**



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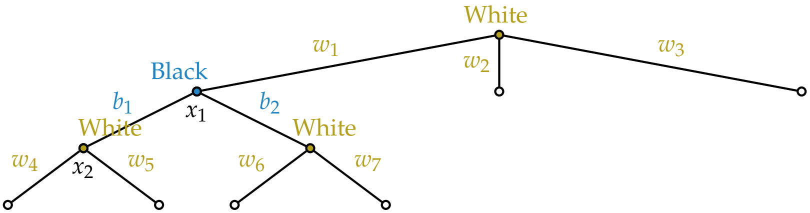

Strategy: mapping from **game situation** to action, i.e., what action to take at every vertex of this game tree

a complete plan to play the game at every game situation

What is a strategy? (contd.)

Definition (Strategy)

A **strategy** for **W** is a function s_W that associates every game situation $(x_0, x_1, x_2, \ldots, x_k) \in H$ (set of all game situations), k even, with a board position x_{k+1} such that the move $x_k \to x_{k+1}$ is a single valid move for W.

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- Similar definition of s_B for B.
- Note: A strategy pair (s_W, s_B) determines outcome (also called one play of the game) – a path through the game tree.

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Can a player guarantee an outcome?

Winning/Drawing Strategies

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- Analogous definitions of s_B^* and s_B' for **B**
- Not obvious if such strategies exist.

Contents

An Early Result (von Neumann, 1928)

Theorem

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1. ***W*** *has a winning strategy*

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An Early Result (von Neumann, 1928)

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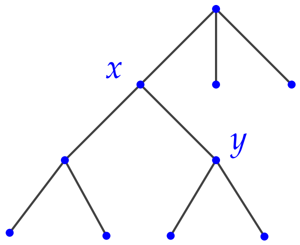
Chess would have been a boring game if any of these answers were known

Setup of the Proof

- Each vertex in the tree represents a game situation, the edges represent actions



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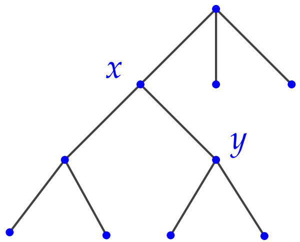

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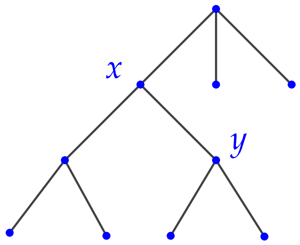

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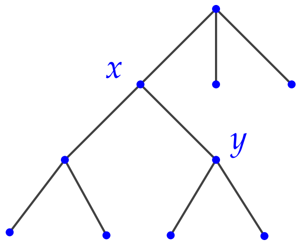

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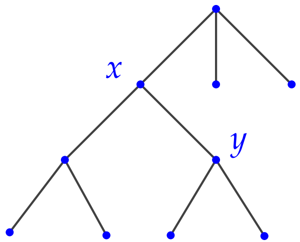
- $\Gamma(y)$ is a subtree of $\Gamma(x)$, $n_y < n_x$



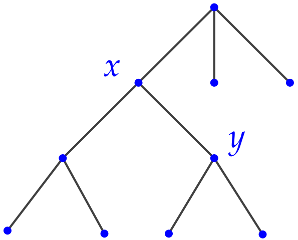
Proof of Chess Theorem

The proof is via induction on n_x.

Question

Does the Theorem hold for $n_x = 1$?

- if **W** king is removed, **B** wins



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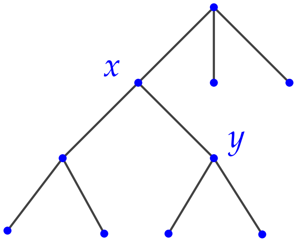

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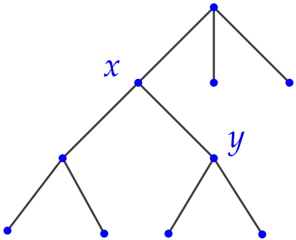
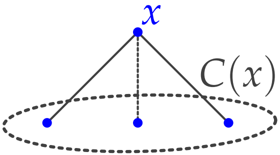
- if both kings present, $n_x = 1$ implies that the game ends in a draw



Extend to $n_x > 1$



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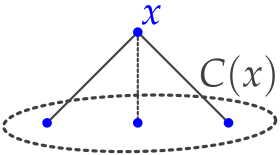

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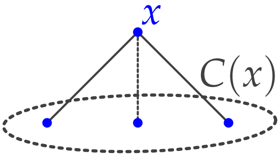

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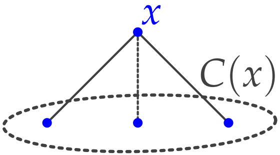

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- Let $C(x)$ denote vertices reachable from x in one move

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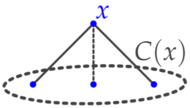
WLOG assume **W** moves at x

- **Case 1:** If $\exists y \in C(x)$ s.t. condition 1 of the theorem is true, then condition 1 is true for x



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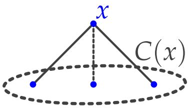

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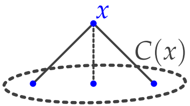

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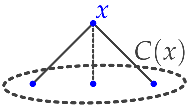
- **Case 3:** Neither case 1 nor case 2 is true



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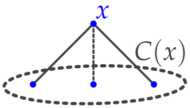

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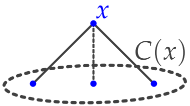

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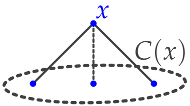

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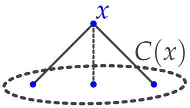

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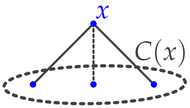
  - Since case 1 does not hold either, **W** cannot guarantee a win in y'



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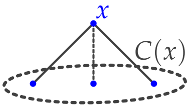
  - Hence **W** picks action to go to y', where **B** can only guarantee a draw (induction hypothesis)



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Indian Institute of Technology Bombay