



भारतीय प्रौद्योगिकी संस्थान मुंबई  
Indian Institute of Technology Bombay

# CS 6001: Game Theory and Algorithmic Mechanism Design

Week 1

Swaprava Nath

Slide preparation acknowledgments: Ramsundar Anandanarayanan and Harshvardhan Agarwal

ज्ञानम् परमम् ध्येयम्

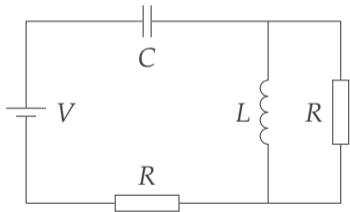
Knowledge is the supreme goal



- ▶ Relation between Game Theory and Mechanism Design
- ▶ What is a Game?
- ▶ An Example Game: Chess
- ▶ Theory of The Game of Chess

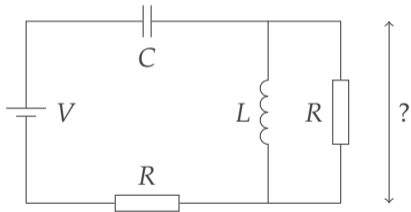


- Circuit **analysis**





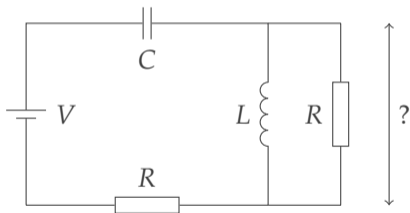
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**analysis**



- Circuit **analysis** and **synthesis**

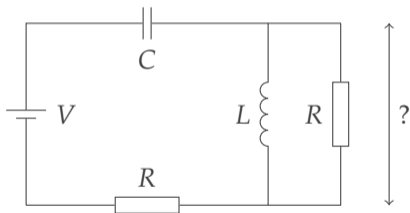


**analysis**

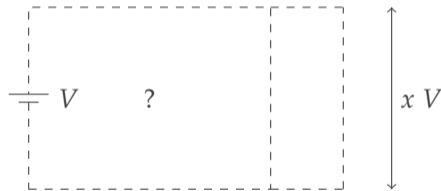




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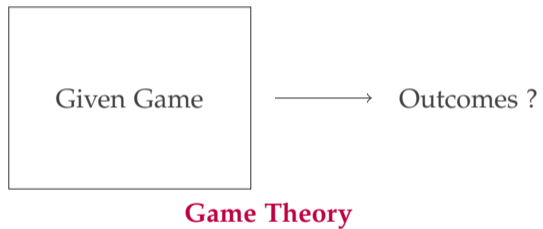
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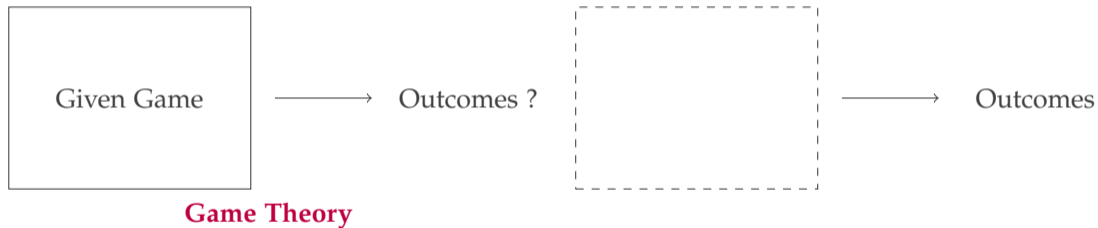
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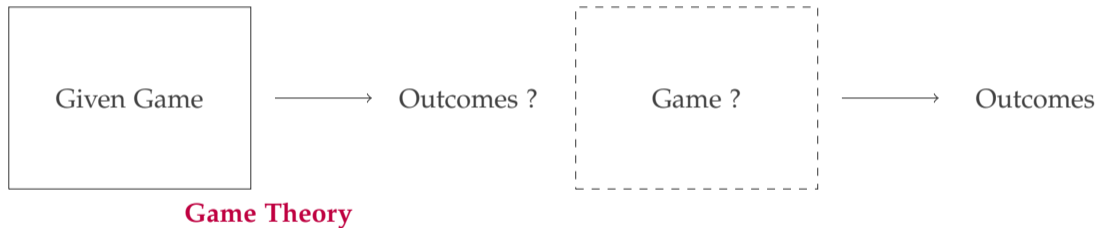
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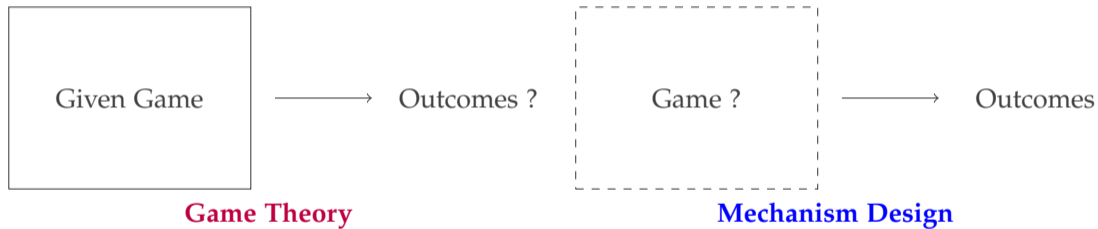
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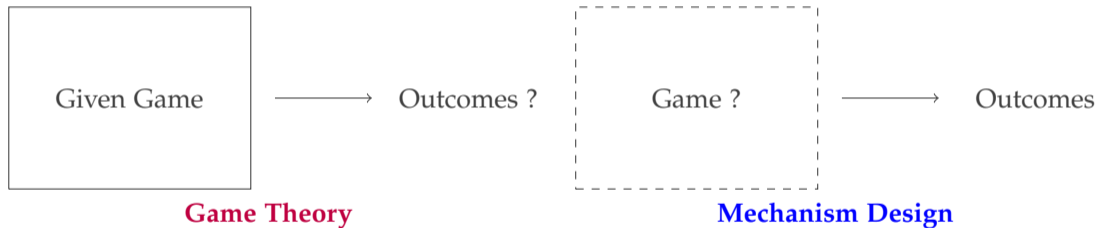
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- Social **analysis** and **synthesis**



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# Game: Neighboring Kingdom's Dilemma



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		Agri	War
Pala	Agri	5,5	0,6
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  - Not in multi-state games
- Games can be of many *kinds* and *representations*:  
**Normal form, Extensive form, Static, Dynamic, Repeated, Stochastic, ...**



## Definition

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- **Goal of game theory: predict** the outcomes of a game (refer to the dilemma game)

# Assumptions of Game Theory



This course is an axiomatic analysis of multi-agent behavior – and the axioms are as follows

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  - ... ad infinitum

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## Question

How does common knowledge percolate?

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Let us think in steps

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## Assumption in Game Theory

The fact that all players are rational and intelligent is a common knowledge



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- ▶ What is a Game?
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- Starts with **W**, players take turns
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  - 1 **Win** for **W**: if W captures B king
  - 2 **Win** for **B**: if B captures W king
  - 3 **Draw**: everything else, e.g., if nobody has legal moves but kings are not in check, both players agree to a draw, board position is such that nobody can win, ...

# Natural Questions from Game Theorist's perspective



## Question

Does **W** have a winning strategy?

i.e., a plan of moves such that it wins **irrespective** of the moves of **B**?

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Or do either have at least a draw guaranteeing strategy?

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- Neither may be possible – not synonymous with the end of the game

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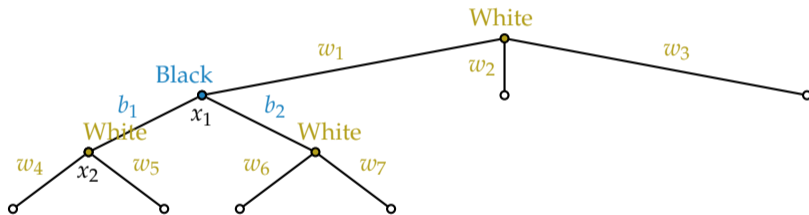


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  - $x_k \rightarrow x_{k+1}$ 
    - $k$  even – created by a single action of W
    - $k$  odd – created by a single action of B



# What is a strategy? (contd.)

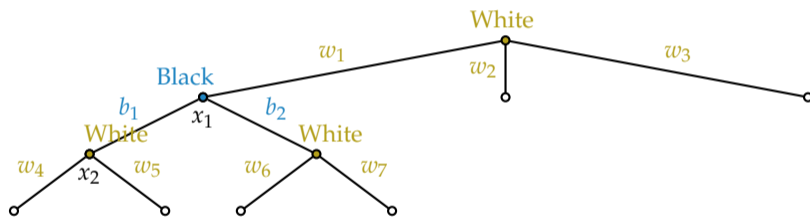
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## What is a strategy? (contd.)

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**Strategy:** mapping from **game situation** to action, i.e., what action to take at every vertex of this game tree

a complete plan to play the game at every game situation

# What is a strategy? (contd.)



## Definition (Strategy)

A **strategy** for **W** is a function  $s_W$  that associates every game situation  $(x_0, x_1, x_2, \dots, x_k) \in H$  (set of all game situations),  $k$  even, with a board position  $x_{k+1}$  such that the move  $x_k \rightarrow x_{k+1}$  is a single valid move for W.

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- Note: A strategy pair  $(s_W, s_B)$  determines **outcome** (also called one play of the game) – a path through the game tree.

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## Question

Can a player guarantee an outcome?



- A **winning strategy** for **W** is a strategy  $s_W^*$  such that for every  $s_B$ ,  $(s_W^*, s_B)$  ends in a win for **W**.

# Winning/Drawing Strategies



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- Not obvious if such strategies exist.



- ▶ Relation between Game Theory and Mechanism Design
- ▶ What is a Game?
- ▶ An Example Game: Chess
- ▶ Theory of The Game of Chess

# An Early Result (von Neumann, 1928)



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*In chess, one and only one of the following statements is true*

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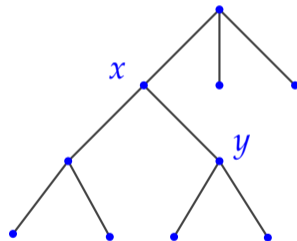
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**Chess would have been a boring game if any of these answers were known**

# Setup of the Proof



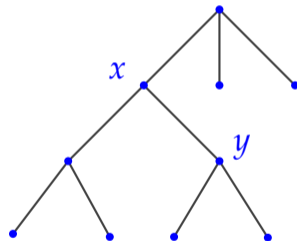
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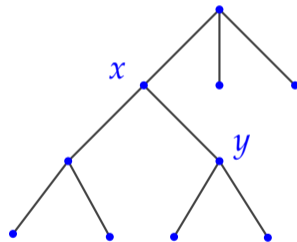
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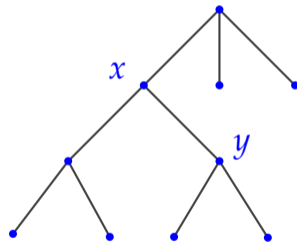
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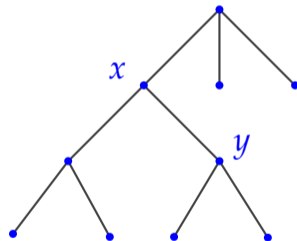
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- $\Gamma(y)$  is a subtree of  $\Gamma(x)$ ,  $n_y < n_x$



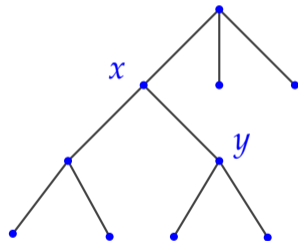
# Proof of Chess Theorem



The proof is via induction on  $n_x$ .

**Question**

Does the Theorem hold for  $n_x = 1$  ?



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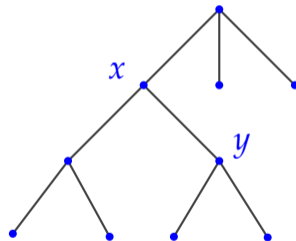


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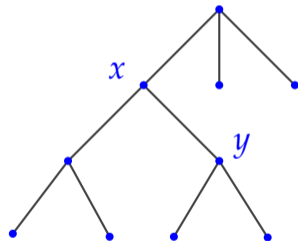


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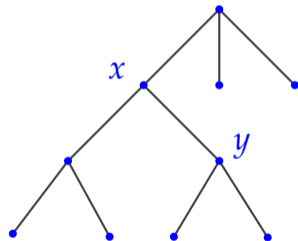


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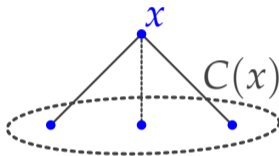
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- if **B** king is removed, **W** wins
- if both kings present,  $n_x = 1$  implies that the game ends in a draw



# Extend to $n_x > 1$

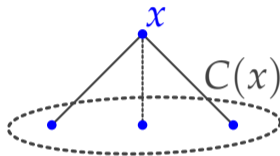


## Notation

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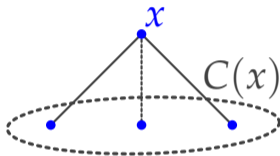
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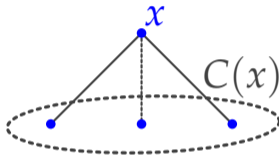
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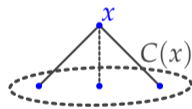
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- Let  $C(x)$  denote vertices reachable from  $x$  in one move



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WLOG assume **W** moves at  $x$

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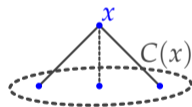




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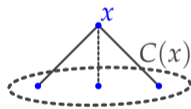




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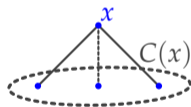




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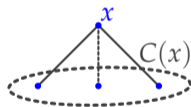




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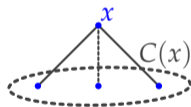




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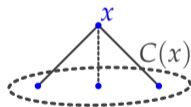




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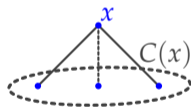




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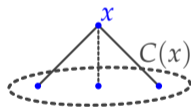




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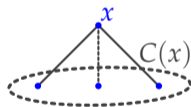




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  - Hence **W** picks action to go to  $y'$ , where **B** can only guarantee a draw (induction hypothesis)



# Not always the case



## Halving game

- Two player game

# Not always the case



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- Start with a number,  $n \rightarrow \text{currNumber}$

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भारतीय प्रौद्योगिकी संस्थान मुंबई  
**Indian Institute of Technology Bombay**