भारतीय प्रौद्योगिकी संस्थान मुंबई

Indian Institute of Technology Bombay

CS 6001: Game Theory and Algorithmic Mechanism Design

Week 1

Swaprava Nath

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ज्ञानम् परमम् ध्येयम्

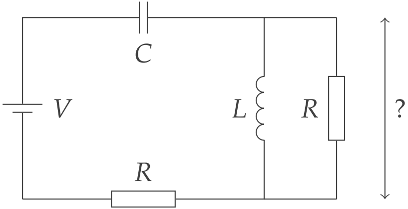
Knowledge is the supreme goal

Contents

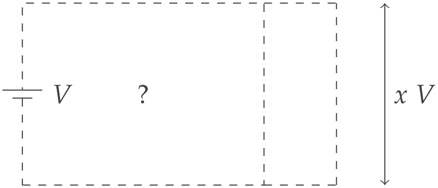
- Relation between Game Theory and Mechanism Design
- What is a Game?
- An Example Game: Chess
- Theory of The Game of Chess

Typical Engineering Courses

- Circuit **analysis** and **synthesis**



analysis



synthesis

Similarly ...

Game Theory **Mechanism Design**

- Social **analysis** and **synthesis**

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Game: Neighboring Kingdom's Dilemma

Pala (rows) vs Rashtrakuta (columns)

	Agri	War
Agri	5, 5	0, 6
War	6, 0	1, 1

Question

What is a reasonable outcome of this game?

Game

		Rashtrakuta: Agri	Rashtrakuta: War
Pala	Agri	5, 5	0, 6
Pala	War	6, 0	1, 1

- A **Game** is a formal representation of the **strategic** interaction between **players**
- The choices available to the players are called **actions**
- The **mapping** from the state of the game to **actions**: **strategy**
  - In single-state games, **strategy** and **action** are equivalent
  - Not in multi-state games
- Games can be of many *kinds* and *representations*:
  Normal form, **Extensive form**, **Static**, **Dynamic**, **Repeated**, **Stochastic**, ...

Game Theory

Definition

Game theory is the formal study of strategic interaction between players, who are **rational** and **intelligent**.

- A player is **rational** if she picks an action to achieve her most desired outcome, i.e., maximize her *happiness*
- A player is **intelligent** if she knows the rules of the game perfectly and can pick (i.e., has that computational ability) the *best* action considering that there are other rational and intelligent players in the game
- **Goal of game theory:** **predict** the outcomes of a game (refer to the dilemma game)

Assumptions of Game Theory

This course is an axiomatic analysis of multi-agent behavior – and the axioms are as follows

- **Rationality**: A player is rational if she picks actions to *maximize* her utility
- **Intelligence**: A player is intelligent if she knows the rules of the game **perfectly** and picks actions considering that there are other *rational* and *intelligent* players.
- **Common Knowledge** (CK): A **fact** is common knowledge if
  - all players know the **fact**
  - all players know that all players know the **fact**
  - all players know that all players know that all players know the **fact**
  - ... ad infinitum

Implication of CK: Blue-eyed islander problem

- **Location**: an isolated island (does not have any reflecting device)
- Three men live on this island (their eye colors can be either **blue** or **black** – but they never talk about their eye colors)
- One day an *all knowing* sage[1] comes in and says: "Blue-eyed people are bad for the island and must leave. There is at least one blue-eyed person on this island."
- **Consequence**: if someone realizes if his eye color is blue, he must leave at the end of the day

Question

How does common knowledge percolate?

[1] This person is correct beyond any question. Whatever he says must be true.

Percolation of Common Knowledge

Let us think in steps

- If there was **one** blue-eyed man
  - he would see the other two have black eyes
  - sage is always correct, he must be the only blue-eyed person
  - leaves at end of day 1
  - seeing him leave, the other two men realize that their eye color is black, stays back
- If there were **two** blue-eyed men
  - each of them would see one blue and one black
  - if there was only one, then by the previous argument, he should have left after day 1
  - when that does not happen, then on day 2, both blue-eyed men realizes their eye color, leaves by the end of the day
- If there are **three** blue-eyed men, use the same argument to conclude that all of them leave at the end of day 3

Assumption in Game Theory

The fact that all players are rational and intelligent is a common knowledge

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Game of Chess

Description

- Two-player game: White (**W**) and Black (**B**) – 16 pieces each
- Every piece has some legal moves – **actions**
- Starts with **W**, players take turns
- Ends in
  1. **Win** for **W**: if W captures B king
  2. **Win** for **B**: if B captures W king
  3. **Draw**: everything else, e.g., if nobody has legal moves but kings are not in check, both players agree to a draw, board position is such that nobody can win, ...

Natural Questions from Game Theorist's perspective

Question

Does **W** have a winning strategy?
i.e., a plan of moves such that it wins **irrespective** of the moves of **B**?

Question

Does **B** have a winning strategy?

Question

Or do either have at least a draw guaranteeing strategy?

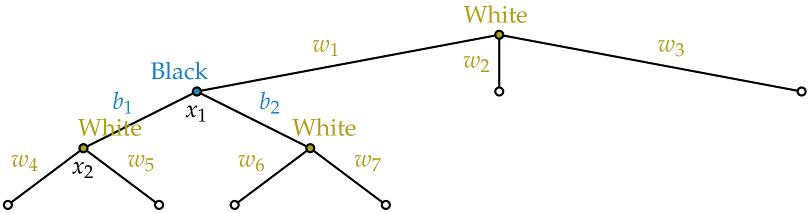
- Neither may be possible – not synonymous with the end of the game

What is a strategy?

- In the context of chess, **board position** is different from **game situation**
- More than one sequence of moves can bring to the same board position.
- Denote a board position by x_k
- **Game situation** is a finite sequence $(x_0, x_1, x_2, \ldots, x_k)$ of board positions such that:
  - x_0 is the opening board position
  - $x_k \to x_{k+1}$
    k even – created by a single action of W
    k odd – created by a single action of B

What is a strategy? (contd.)

Board positions may repeat in this tree, but a vertex is unique – **game situation**



Strategy: mapping from **game situation** to action, i.e., what action to take at every vertex of this game tree

a complete contingency plan

What is a strategy? (contd.)

Definition (Strategy)

A **strategy** for **W** is a function s_W that associates every game situation $(x_0, x_1, x_2, \ldots, x_k) \in H$ (set of all game situations), k even, with a board position x_{k+1} such that the move $x_k \to x_{k+1}$ is a single valid move for W.

- Similar definition of s_B for B.
- Note: A strategy pair (s_W, s_B) determines outcome (also called one play of the game) – a path through the game tree.

Question

Is this a finite game? Where does it end?

Question

Can a player guarantee an outcome?

Winning/Drawing Strategies

- A winning strategy for **W** is a strategy s_W^* such that for every s_B, (s_W^*, s_B) ends in a win for **W**.
- A strategy guaranteeing at least a draw for **W** is a strategy s'_W such that for every s_B, (s'_W, s_B) either ends in a draw or a win for **W**.
- Analogous definitions of s_B^* and s'_B for **B**
- Not obvious if such strategies exist.

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An Early Result (von Neumann, 1928)

Theorem

In chess, one and only one of the following statements is true

1. ***W*** *has a winning strategy*
2. ***B*** *has a winning strategy*
3. *Each player has a draw guaranteeing strategy*

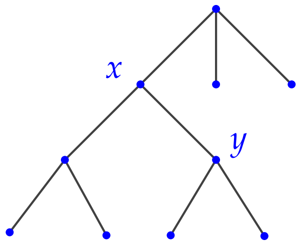
Awesome but not enough

- These options are not exhaustive, e.g., nothing could be guaranteed
- The theorem **does not** say what that strategy is
- It is not known: *which one* is true and *what* is that strategy

Chess would have been a boring game if any of these answers were known

Setup of the Proof

- Each vertex in the tree represents a game situation, the edges represent actions
- $\Gamma(x)$: Subtree rooted at x (including itself)
- n_x : Number of vertices in subtree $\Gamma(x)$
- In the graph, y is a vertex in $\Gamma(x), y \neq x$.
- $\Gamma(y)$ is a subtree of $\Gamma(x)$, $n_y < n_x$



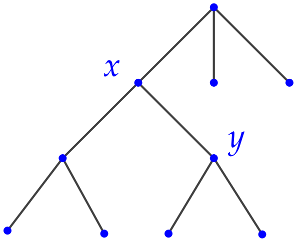
Proof of Chess Theorem

The proof is via induction on n_x.

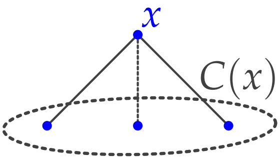
Question

Does the Theorem hold for $n_x = 1$?

- if **W** king is removed, **B** wins
- if **B** king is removed, **W** wins
- if both kings present, $n_x = 1$ implies that the game ends in a draw



Extend to $n_x > 1$



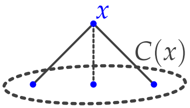
Notation

- Suppose x is a vertex with $n_x > 1$
- **Induction hypothesis**: for all vertices $y \neq x$ such that $\Gamma(y)$ is a subgame of $\Gamma(x)$, the theorem holds
- Then we show that the theorem holds for x as well
- Let $C(x)$ denote vertices reachable from x in one move

Extend to $n_x > 1$

WLOG assume **W** moves at x

- **Case 1:** If $\exists y \in C(x)$ s.t. condition 1 of the theorem is true, then condition 1 is true for x
- **W** picks that move which moves the game to y
- **Case 2:** If $\forall y \in C(x)$, condition 2 is true, then every move by white leads to **B** winning the game. Hence, condition 2 is true for x.
- **Case 3:** Neither case 1 nor case 2 is true
  - Case 1 does not hold
  - **W** does not have a winning strategy in any $y \in C(x)$, since induction hypothesis holds for every $y \in C(x)$, either **B** has winning strategy or both have draw-guaranteeing strategy.
  - Case 2 does not hold either
  - This implies $\exists y' \in C(x)$ s.t. **B** does not have a winning strategy
  - Since case 1 does not hold either, **W** cannot guarantee a win in y'
  - Hence **W** picks action to go to y', where **B** can only guarantee a draw (induction hypothesis)



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