CS 6001: Game Theory and Algorithmic Mechanism Design

Week 2

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Slide preparation acknowledgments: Ramsundar Anandanarayanan and Harshvardhan Agarwal

ज्ञानम् परमम् ध्येयम्
Knowledge is the supreme goal

Contents

Normal Form Games

- It is a representation technique for games – particularly suitable for **static games**
- In a *static game*, the players interact only once with each other

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- **Normal form** representation is a tuple $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$
- If S_i is finite $\forall i \in N$, this is called a finite game.

Example: Penalty Shootout Game

		Goalkeeper		
		L	C	R
Shooter	L	$-1, 1$	$1, -1$	$1, -1$
	C	$1, -1$	$-1, 1$	$1, -1$
	R	$1, -1$	$1, -1$	$-1, 1$

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- $N = \{1, 2\}, 1 = \text{Shooter}, 2 = \text{Goalkeeper}$
- $S_1 = S_2 = \{L, C, R\}$
- $u_1(L, L) = -1, u_1(L, C) = u_1(L, R) = 1$
- $u_2(L, L) = 1, u_2(L, C) = u_2(L, R) = -1$
- (loosely) $u_1(X, X) = -1 = -u_2(X, X), u_1(X, Y) = -u_2(X, Y) = 1$

Contents

Domination in NFGs

		Player 2		
		L	C	R
Player 1	U	1, 0	1, 3	3, 2
	D	−1, 6	0, 5	3, 3

Domination in NFGs

		Player 2		
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Question

Will a **rational** Player 2 ever play R?

Dominated Strategy

Definition (Strictly Dominated Strategy)

A strategy $s'_i \in S_i$ of player i is **strictly dominated** if there exists another strategy $s_i \in S_i$ such that **for every strategy profile** $s_{-i} \in S_{-i}$ of the other players, $u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$.

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Definition (Weakly Dominated Strategy)

A strategy $s_i' \in S_i$ of player i is **weakly dominated** if there exists another strategy $s_i \in S_i$ such that **for every strategy profile** $s_{-i} \in S_{-i}$ of the other players $u_i(s_i, s_{-i}) \geqslant u_i(s_i', s_{-i})$ and **there exists some** $\tilde{s}_{-i} \in S_{-i}$ such that $u_i(s_i, \tilde{s}_{-i}) > u_i(s_i', \tilde{s}_{-i})$.

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This $\tilde{s}_{-i}$ is specific to the pair of strategies (s_i, s_i'), i.e., $\tilde{s}_{-i} = \tilde{s}_{-i}(s_i, s_i')$

Dominated Strategy

		Player 2		
		L	C	R
Player 1	U	1, 0	1, 3	3, 2
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Strictly / Weakly dominated strategy?

Dominated Strategy

		Player 2		
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Player 1	U	1,0	1,3	3,2
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Strictly / Weakly dominated strategy?

R is strictly dominated (by C) while D is weakly dominated (by U)

Dominant Strategy

A strategy s_i' can be dominated by s_i, and a different strategy s_i'' can be dominated by $\tilde{s}_i$

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Examples of **dominant strategy**

- Neighboring kingdom's dilemma
- Indivisible item for sale

Neighboring Kingdom’s Dilemma

		Rashtrakuta	
		Agri	War
Pala	Agri	5,5	0,6
	War	6,0	1,1

Neighboring Kingdom's Dilemma

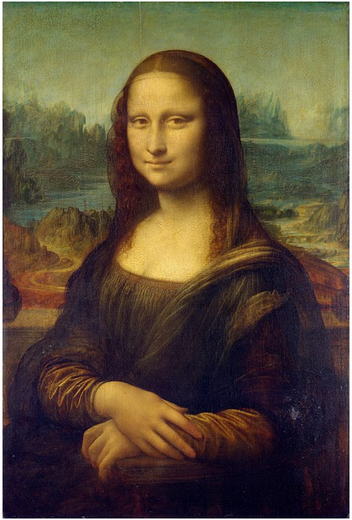
		Rashtrakuta	
		Agri	War
Pala	Agri	5, 5	0, 6
	War	6, 0	1, 1

Question

Is there a dominant strategy in this game? Which kind?

Indivisible Item for Sale

- Two players value an indivisible item as v_1 and v_2 respectively



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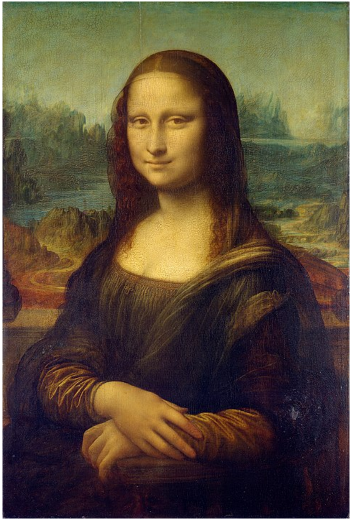
- Each player's action: a number in $[0, M]$, $M \gg v_1, v_2$



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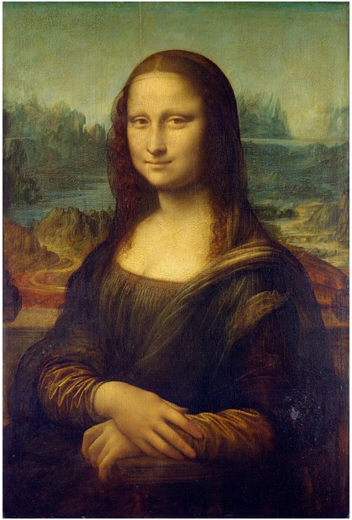

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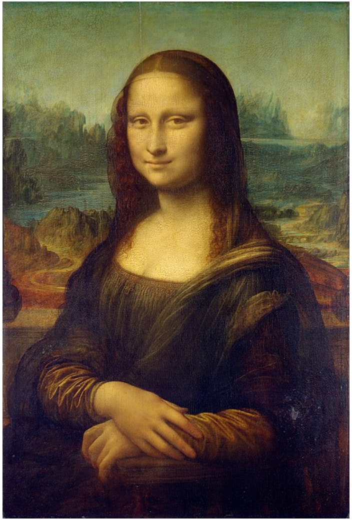

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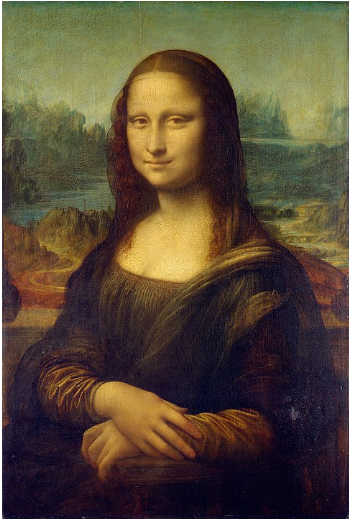
- utility of losing player = 0



Indivisible Item for Sale

Normal form representation of the game

- $N = \{1,2\}, S_1 = S_2 = [0, M]$
- Agents pick s_i, while their **real** value for the item is v_i, and s_i may **not** be the same as v_i

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$$u_1(s_1, s_2) = \begin{cases} v_1 - s_2 & \text{if } s_1 \geqslant s_2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$u_2(s_1, s_2) = \begin{cases} v_2 - s_1 & \text{if } s_1 < s_2 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

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Definition (Weak Domination)

A strategy $s_i \in S_i$ of player i weakly dominates $s_i' \in S_i$ if **for every strategy profile** $s_{-i} \in S_{-i}$ of the other players $u_i(s_i, s_{-i}) \geqslant u_i(s_i', s_{-i})$ and **there exists some** $\tilde{s}_{-i} \in S_{-i}$ such that $u_i(s_i, \tilde{s}_{-i}) > u_i(s_i', \tilde{s}_{-i})$. $[\tilde{s}_{-i} = \tilde{s}_{-i}(s_i, s_i')]$

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Dominant Strategy Equilibrium

Definition (Dominant Strategy Equilibrium)

A strategy profile $(s_1^*, s_2^*, \ldots, s_n^*)$ is a strictly (weakly) dominant strategy equilibrium (SDSE/WDSE) if s_i^* is strictly (weakly) dominant strategy $\forall i \in N$.

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Example of **dominant strategy equilibrium**

Player 2

		D	E
Player 1	A	5,5	0,5
	B	5,0	1,1
	C	4,0	1,1

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Question

What kind of equilibrium in this game?

Existence of Dominant Strategies

Not guaranteed!

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Player 2

		L	R
Player 1	L	1,1	0,0
	R	0,0	1,1

Co-ordination game

Existence of Dominant Strategies

Not guaranteed!

		Player 2	
		L	R
Player 1	L	1, 1	0, 0
	R	0, 0	1, 1

Co-ordination game

		Friend 2	
		F	C
Friend 1	F	2, 1	0, 0
	C	0, 0	1, 2

Football or Cricket Game

Existence of Dominant Strategies

Not guaranteed!

		Player 2	
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Player 1	L	1, 1	0, 0
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Co-ordination game

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Football or Cricket Game

If **dominance** cannot explain a reasonable outcome – refine equilibrium concept

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Nash Equilibrium (Nash 1951)

No player gains by a unilateral deviation

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Definition (Nash Equilibrium)

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		F	C
Friend 1	F	2,1	0,0
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Football or Cricket Game

Best Response View

- A best response of a player i against the strategy profile s_{-i} of other players is a strategy that gives the maximum utility i.e.,

$$B_i(s_{-i}) = \{s_i \in S_i : u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}), \forall s_i' \in S_i\}$$

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Relationship between SDSE, WDSE, PSNE?

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- PSNE is a strategy profile (s^*_i, s^*_{-i}) s.t.

$$s^*_i \in B_i(s^*_{-i}), \forall i \in N$$

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Question

Relationship between SDSE, WDSE, PSNE?

Answer

SDSE $\implies$ WDSE $\implies$ PSNE

How to find equilibrium?

- Rational players do not play **dominated strategies**

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Player 2

Player 1

	L	C	R
T	1,2	2,3	0,3
M	2,2	2,1	3,2
B	2,1	0,0	1,0

How to find equilibrium?

- Rational players do not play **dominated strategies**
- To obtain rational outcomes eliminate dominated strategies
- For strictly dominated strategies the order of elimination does not matter **(homework)**
- It matters for weakly dominated strategies – some reasonable outcomes are also eliminated

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		L	C	R
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- Order T, R, B, C $\rightarrow (M, L) : (2, 2)$

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- Order T, R, B, C → $(M, L) : (2, 2)$
- Order B, L, C, T → $(M, R) : (3, 2)$

Contents

Risk Aversion of Players

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What if the other player does not pick an equilibrium action (Nash)?

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What if the other player does not pick an equilibrium action (Nash)?

Picking T is less risky for player 1

Max-min Strategy

Definition

The worst case optimal choice is **max-min strategy**

$$u_i(s_i, s_{-i})$$

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Max-min value (utility at the max-min strategy) of player i is given by

$$\underline{v}_i = \max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$$

$$u_i(s_i^{\max\min}, t_{-i}) \geqslant \underline{v}_i, \qquad \forall t_{-i} \in S_{-i}$$

Max-min and Dominant Strategies

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If s_i^* is ***dominant strategy*** *for player i, then it is a* ***max-min strategy*** *for player i as well.*

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Let s_i^* be dominant strategy for player i

$$u_i(s_i^*, s_{-i}) \geqslant u_i(s_i', s_{-i}), \ \forall s_{-i} \in S_{-i}, \forall s_i' \in S_i \tag{5}$$

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Every ***PSNE*** $s^* = (s_1^*, s_2^*, \ldots, s_n^*)$ *of a* normal form game *satisfies* $u_i(s^*) \geqslant \underline{v}_i, \forall i \in N.$

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☐

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Question

What happens to stability and security when some strategies are eliminated?

Iterated elimination of dominated strategies (contd.)

		Player 2		
		L	C	R
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Does it change the maxmin value?

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Consider in the above example: elimination of dominated strategy B for player 1

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Maxmin values	Player 1	Player 2
Before	2	0
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Maxmin value is not affected for the player whose **dominated strategy** is removed

A Result for Iterated Elimination

Theorem

Consider an NFG $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$*, and let* $s'_j \in S_j$ *be a dominated strategy. Let* G' *be the residual game after removing* s'_j*. Then, the maxmin value of* j *in* G' *is equal to her maxmin value in* G*.*

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Intuition

- Maxmin is the 'max' of all 'min's
- Elimination affects one 'min'
- But that does not affect the 'max' since the strategy was dominated

Proof

Maxmin value of player j in G $\underline{v}_j = \max_{s_j \in S_j} \min_{s_{-j} \in S_{-j}} u_j(s_j, s_{-j})$

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Suppose t_j dominates s'_j in $G, t_j \in S_j \setminus \{s'_j\},$ then, $u_j(t_j, s_{-j}) \geqslant u_j(s'_j, s_{-j}), \forall s_{-j} \in S_{-j}$

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Proof (contd.)

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Proof (contd.)

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Consider G and $\hat{G}$ are games before and after elimination of a strategy (not necessarily dominated). If s^* is a PSNE in G and survives in $\hat{G}$, then s^* is a PSNE in $\hat{G}$ too.

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Intuition

PSNE was a maxima of utility of i among the strategies of i. Removing other strategies does not affect maximality.

Proof: exercise.

Can new equilibrium be generated?

Theorem

Consider NFG G. Let $\hat{s}_j$ be a weakly dominated strategy of j. If $\hat{G}$ is obtained from G eliminating $\hat{s}_j$, then every PSNE of $\hat{G}$ is a PSNE of G.

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But old PSNEs could be killed: saw in the previous example

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- Elimination of weakly dominated strategy may reduce the set of PSNEs, but never adds new
- The maxmin values of the player whose strictly or weakly dominated strategies are removed remain unaffected

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Matrix games: *two player zero-sum* games

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Why called **matrix** game?

Answer

Possible to represent the game with only one matrix considering the utilities of player 1; player 2's utilities are negative of this matrix

Example: Penalty shoot game

		Player 2	
		L	R
Player 1	L	$-1, 1$	$1, -1$
	R	$1, -1$	$-1, 1$

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$$\Longrightarrow \qquad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} =: u$$

		L	*R*	*maxmin*
Player 1	*L*	−1	1	−1
	R	1	−1	−1
	minmax	1	1	

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Player 2's ***maxmin*** *value is the negative of the* ***minmax*** *value of this matrix*

		L	R	$maxmin$
Player 1	L	−1	1	−1
	R	1	−1	−1
	$minmax$	1	1	

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