भारतीय प्रौद्योगिकी संस्थान मुंबई
Indian Institute of Technology Bombay

CS 6001: Game Theory and Algorithmic Mechanism Design

Week 2

Swaprava Nath

Slide preparation acknowledgments: Ramsundar Anandanarayanan and Harshvardhan Agarwal

ज्ञानम् परमम् ध्येयम्
Knowledge is the supreme goal

Contents

Normal Form Games

- It is a representation technique for games – particularly suitable for **static games**
- In a *static game*, the players interact only once with each other

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- **Normal form** representation is a tuple $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$
- If S_i is finite $\forall i \in N$, this is called a finite game.

Example: Penalty Shoot Game

		Goalkeeper		
		L	C	R
Shooter	L	$-1, 1$	$1, -1$	$1, -1$
	C	$1, -1$	$-1, 1$	$1, -1$
	R	$1, -1$	$1, -1$	$-1, 1$

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- $N = \{1, 2\}, 1 = \text{Shooter}, 2 = \text{Goalkeeper}$

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Shooter	L	$-1, 1$	$1, -1$	$1, -1$
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- $N = \{1, 2\}, 1 = \text{Shooter}, 2 = \text{Goalkeeper}$
- $S_1 = S_2 = \{L, C, R\}$
- $u_1(L, L) = -1, u_1(L, C) = u_1(L, R) = 1$
- $u_2(L, L) = 1, u_2(L, C) = u_2(L, R) = -1$
- (loosely) $u_1(X, X) = -1 = -u_2(X, X), u_1(X, Y) = -u_2(X, Y) = 1$

Contents

Domination in NFGs

Player 2

		L	C	R
Player 1	U	1, 0	1, 3	3, 2
	D	−1, 6	0, 5	3, 3

Domination in NFGs

Player 2

		L	C	R
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Question

Will a **rational** Player 2 ever play R?

Dominated Strategy

Definition (Strictly Dominated Strategy)

A strategy $s_i' \in S_i$ of player i is **strictly dominated** if there exists another strategy $s_i \in S_i$ such that **for every strategy profile** $s_{-i} \in S_{-i}$ of the other players, $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$.

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Definition (Weakly Dominated Strategy)

A strategy $s'_i \in S_i$ of player i is **weakly dominated** if there exists another strategy $s_i \in S_i$ such that **for every strategy profile** $s_{-i} \in S_{-i}$ of the other players $u_i(s_i, s_{-i}) \geqslant u_i(s'_i, s_{-i})$ and **there exists some** $\tilde{s}_{-i} \in S_{-i}$ such that $u_i(s_i, \tilde{s}_{-i}) > u_i(s'_i, \tilde{s}_{-i})$.

Dominated Strategy

		Player 2		
		L	C	R
Player 1	U	$1, 0$	$1, 3$	$3, 2$
	D	$-1, 6$	$0, 5$	$3, 3$

Dominated Strategy

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Player 1	U	1,0	1,3	3,2
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Strictly / Weakly dominated strategy?

Dominated Strategy

		Player 2		
		L	C	R
Player 1	U	1, 0	1, 3	3, 2
	D	−1, 6	0, 5	3, 3

Strictly / Weakly dominated strategy?

R is strictly dominated (by C) while D is weakly dominated (by U)

Dominant Strategy

A strategy s_i' can be dominated by s_i, and a different strategy s_i'' can be dominated by $\tilde{s}_i$

Definition (Dominant Strategy)

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Examples of **dominant strategy**

- Neighboring kingdom's dilemma
- Indivisible item for sale

Neighboring Kingdom’s Dilemma

		Rashtrakuta	
		Agri	War
Pala	Agri	5,5	0,6
	War	6,0	1,1

Neighboring Kingdom’s Dilemma

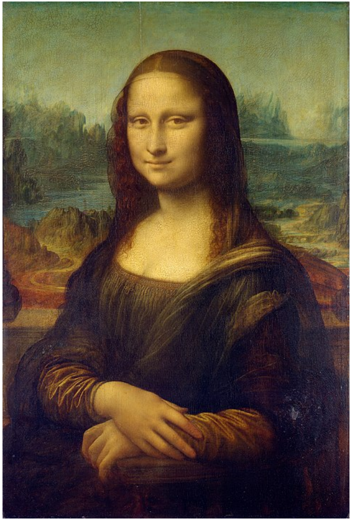
		Rashtrakuta	
		Agri	War
Pala	Agri	5,5	0,6
	War	6,0	1,1

Question

Is there a dominant strategy in this game? Which kind?

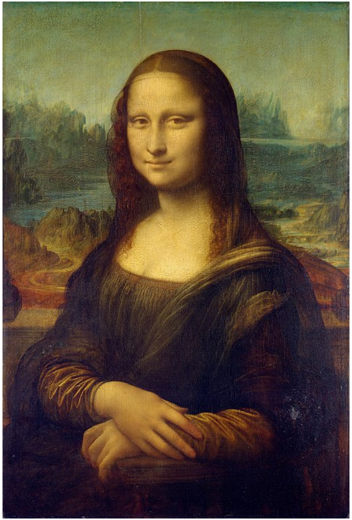
Indivisible Item for Sale

- Two players value an indivisible item as v_1 and v_2 respectively



Indivisible Item for Sale

- Two players value an indivisible item as v_1 and v_2 respectively
- Each player's action: a number in $[0, M]$, $M \gg v_1, v_2$



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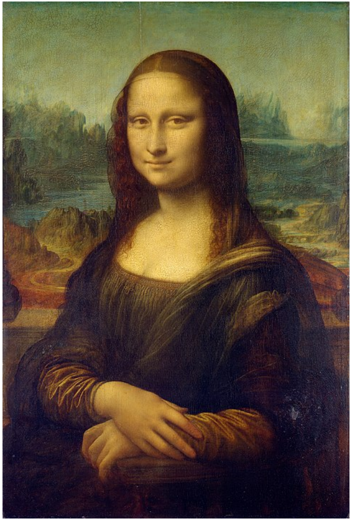

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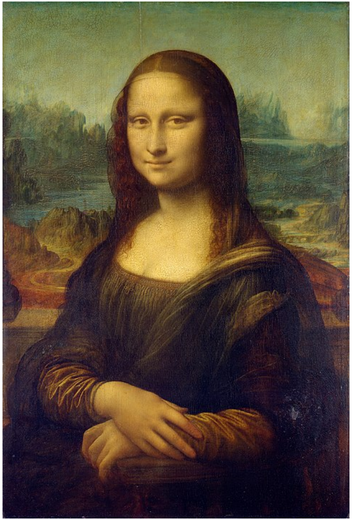

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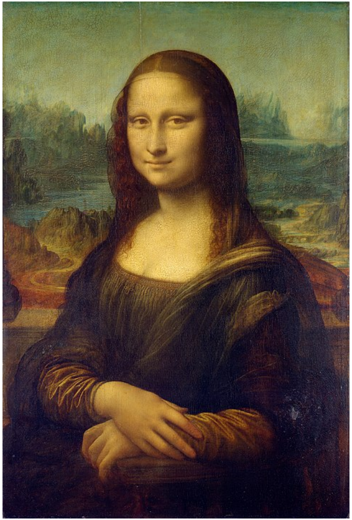
- utility of losing player = 0



Indivisible Item for Sale

Normal form representation of the game

- $N = \{1, 2\}$, $S_1 = S_2 = [0, M]$
- Agents pick s_i, while their **real** value for the item is v_i, and s_i may **not** be the same as v_i

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$$u_1(s_1, s_2) = \begin{cases} v_1 - s_2 & \text{if } s_1 \geqslant s_2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$u_2(s_1, s_2) = \begin{cases} v_2 - s_1 & \text{if } s_1 < s_2 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

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Weakly Dominant Strategy of Second Price Auction

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Definition (Weak Domination)

A strategy $s_i \in S_i$ of player i weakly dominates $s_i' \in S_i$ if **for every strategy profile** $s_{-i} \in S_{-i}$ of the other players $u_i(s_i, s_{-i}) \geqslant u_i(s_i', s_{-i})$ and **there exists some** $\tilde{s}_{-i} \in S_{-i}$ such that $u_i(s_i, \tilde{s}_{-i}) > u_i(s_i', \tilde{s}_{-i})$. $[\tilde{s}_{-i} = \tilde{s}_{-i}(s_i, s_i')]$

Definition (Dominant Strategy)

A strategy s_i is strictly (weakly) dominant strategy for player i if s_i strictly (weakly) dominates all other strategies $s_i' \in S_i \setminus \{s_i\}$.

Dominant Strategy Equilibrium

Definition (Dominant Strategy Equilibrium)

A strategy profile $(s_1^*, s_2^*, \ldots, s_n^*)$ is a strictly (weakly) dominant strategy equilibrium (SDSE/WDSE) if s_i^* is strictly (weakly) dominant strategy $\forall i \in N$.

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Example of **dominant strategy equilibrium**

		Player 2	
		D	E
Player 1	A	5,5	0,5
	B	5,0	1,1
	C	4,0	1,1

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Example of **dominant strategy equilibrium**

Player 2

Player 1		D	E
	A	5, 5	0, 5
	B	5, 0	1, 1
	C	4, 0	1, 1

Question

What kind of equilibrium in this game?

Existence of Dominant Strategies

Not guaranteed!

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		Player 2	
		L	R
Player 1	L	1, 1	0, 0
	R	0, 0	1, 1

Co-ordination game

Existence of Dominant Strategies

Not guaranteed!

		Player 2	
		L	R
Player 1	L	1, 1	0, 0
	R	0, 0	1, 1

Co-ordination game

		Friend 2	
		F	C
Friend 1	F	2, 1	0, 0
	C	0, 0	1, 2

Football or Cricket Game

Existence of Dominant Strategies

Not guaranteed!

		Player 2	
		L	R
Player 1	L	1, 1	0, 0
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Co-ordination game

		Friend 2	
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Football or Cricket Game

If **dominance** cannot explain a reasonable outcome – refine equilibrium concept

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Nash Equilibrium (Nash 1951)

No player gains by a unilateral deviation

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		Friend 2	
		F	C
Friend 1	F	2,1	0,0
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Football or Cricket Game

Best Response View

- A best response of a player i against the strategy profile s_{-i} of other players is a strategy that gives the maximum utility i.e.,

$$B_i(s_{-i}) = \{s_i \in S_i : u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}), \forall s_i' \in S_i\}$$

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- PSNE gives stability – once there, no rational player unilaterally deviates

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Question

Relationship between SDSE, WDSE, PSNE?

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Question

Relationship between SDSE, WDSE, PSNE?

Answer

SDSE $\implies$ WDSE $\implies$ PSNE

How to find equilibrium?

- Rational players do not play **dominated strategies**

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		Player 2		
		L	C	R
Player 1	T	1,2	2,3	0,3
	M	2,2	2,1	3,2
	B	2,1	0,0	1,0

- Order T, R, B, C $\rightarrow (M, L) : (2, 2)$

How to find equilibrium?

- Rational players do not play **dominated strategies**
- To obtain rational outcomes eliminate dominated strategies
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		Player 2		
		L	C	R
Player 1	T	1, 2	2, 3	0, 3
	M	2, 2	2, 1	3, 2
	B	2, 1	0, 0	1, 0

- Order T, R, B, C $\to (M, L) : (2, 2)$
- Order B, L, C, T $\to (M, R) : (3, 2)$

Contents

Risk Aversion of Players

		Player 2	
		L	R
Player 1	T	$2, 1$	$1, -20$
	M	$3, 0$	$-10, 1$
	B	$-100, 2$	$3, 3$

Risk Aversion of Players

		Player 2	
		L	R
Player 1	T	$2, 1$	$1, -20$
	M	$3, 0$	$-10, 1$
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Question

What if the other player does not pick an equilibrium action (Nash)?

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Question

What if the other player does not pick an equilibrium action (Nash)?

Picking T is less risky for player 1

Max-min Strategy

Definition

The worst case optimal choice is **max-min strategy**

$$u_i(s_i, s_{-i})$$

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Max-min value (utility at the max-min strategy) of player i is given by

$$\underline{v}_i = \max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$$

$$u_i(s_i^{\max\min}, t_{-i}) \geqslant \underline{v}_i, \qquad \forall t_{-i} \in S_{-i}$$

Max-min and Dominant Strategies

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*If s_i^* is **dominant strategy** for player i, then it is a **max-min strategy** for player i as well.*

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Let s_i^* be dominant strategy for player i

$$u_i(s_i^*, s_{-i}) \geqslant u_i(s_i', s_{-i}),\ \forall s_{-i} \in S_{-i}, \forall s_i' \in S_i \tag{5}$$

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Every **PSNE** $s^* = (s_1^*, s_2^*, \ldots, s_n^*)$ *of a* normal form game *satisfies* $u_i(s^*) \geqslant \underline{v}_i, \forall i \in N$.

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- Dominance cannot explain all outcomes; games may not have dominant strategies
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Question

What happens to stability and security when some strategies are eliminated?

Iterated elimination of dominated strategies (contd.)

Player 2

		L	C	R
Player 1	T	1, 2	2, 3	0, 3
	M	2, 2	2, 1	3, 2
	B	2, 1	0, 0	1, 0

Iterated elimination of dominated strategies (contd.)

		Player 2		
		L	C	R
Player 1	T	1,2	2,3	0,3
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- Order T, R, B, C → $(M, L) : (2, 2)$

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- Order B, L, C, T → $(M, R) : (3, 2)$

Iterated elimination of dominated strategies (contd.)

Player 2

Player 1		L	C	R
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Does it change the maxmin value?

Iterated elimination of dominated strategies (contd.)

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Consider in the above example: elimination of dominated strategy B for player 1

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Maxmin values	Player 1	Player 2
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Maxmin values	Player 1	Player 2
Before	2	0
After	2	2

Maxmin value is not affected for the player whose **dominated strategy** is removed

A Result for Iterated Elimination

Theorem

Consider an NFG $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$*, and let* $s'_j \in S_j$ *be a dominated strategy. Let* G' *be the residual game after removing* s'_j*. Then, the maxmin value of* j *in* G' *is equal to her maxmin value in* G*.*

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Intuition

- Maxmin is the 'max' of all 'min's
- Elimination affects one 'min'
- But that does not affect the 'max' since the strategy was dominated

Proof

Maxmin value of player j in G $\underline{v}_j = \max_{s_j \in S_j} \min_{s_{-j} \in S_{-j}} u_j(s_j, s_{-j})$

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Suppose t_j dominates s'_j in $G, t_j \in S_j \setminus \{s'_j\},$ then, $\quad u_j(t_j, s_{-j}) \geqslant u_j(s'_j, s_{-j}), \forall s_{-j} \in S_{-j}$

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&= \max_{s_j \in S_j \setminus \{s_j'\}} \min_{s_{-j} \in S_{-j}} u_j(s_j, s_{-j}), \text{ because of the previous inequality}
\end{aligned}
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\begin{aligned}
&\underline{v}_j \quad \text{[maxmin value of } j \text{ in } G\text{]} \\
&= \max_{s_j \in S_j} \min_{s_{-j} \in S_{-j}} u_j(s_j, s_{-j}) \\
&= \max \left\{ \max_{s_j \in S_j \setminus \{s'_j\}} \min_{s_{-j} \in S_{-j}} u_j(s_j, s_{-j}), \min_{s_{-j} \in S_{-j}} u_j(s'_j, s_{-j}) \right\} \\
&= \max_{s_j \in S_j \setminus \{s'_j\}} \min_{s_{-j} \in S_{-j}} u_j(s_j, s_{-j}), \text{ because of the previous inequality} \\
&= \underline{v}'_j \quad \text{[maxmin value of } j \text{ in } G'\text{]}
\end{aligned}
$$

Contents

Preservation of PSNE

Question

What happens to existing equilibrium after iterated elimination?

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Consider G and $\hat{G}$ are games before and after elimination of a strategy (not necessarily dominated). If s^* is a PSNE in G and survives in $\hat{G}$, then s^* is a PSNE in $\hat{G}$ too.

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Intuition

PSNE was a maxima of utility of i among the strategies of i. Removing other strategies does not affect maximality.
Proof: exercise.

Can new equilibrium be generated?

Theorem

Consider NFG G. Let $\hat{s}_j$ *be a weakly dominated strategy of* j*. If* $\hat{G}$ *is obtained from G eliminating* $\hat{s}_j$*, then every PSNE of* $\hat{G}$ *is a PSNE of G.*

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But old PSNEs could be killed: saw in the previous example

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Since s^* is a PSNE in $\hat{G}$ and $t_j \in \hat{S}_j$

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- Elimination of weakly dominated strategy may reduce the set of PSNEs, but never adds new
- The maxmin values of the player whose strictly or weakly dominated strategies are removed remain unaffected

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Answer

Possible to represent the game with only one matrix considering the utilities of player 1; player 2's utilities are negative of this matrix

Example: Penalty shoot game

		Player 2	
		L	R
Player 1	L	$-1, 1$	$1, -1$
	R	$1, -1$	$-1, 1$

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Player 2

Player 1	L	R
L	−1, 1	1, −1
R	1, −1	−1, 1

$$\Longrightarrow \qquad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} =: u$$

Player 1

	L	R	*maxmin*
L	−1	1	−1
R	1	−1	−1
minmax	1	1	

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Player 2's ***maxmin*** *value is the* ***minmax*** *value of this matrix*

		L	*R*	*maxmin*
Player 1	*L*	−1	1	−1
	R	1	−1	−1
	minmax	1	1	

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