CS 6001: Game Theory and Algorithmic Mechanism Design

Week 2

Swaprava Nath

Slide preparation acknowledgments: Ramsundar Anandanarayanan and Harshvardhan Agarwal

Contents

Normal Form Games

- It is a representation technique for games – particularly suitable for **static games**
- In a *static game*, the players interact only once with each other

Notation

- $N = \{1, 2, 3, \ldots, n\}$, set of players
- S_i : set of strategies for player i, $\quad s_i \in S_i$
- Set of strategy profiles $S = \times_{i \in N} S_i$
- A strategy profile $s = (s_1, s_2, s_3, \ldots, s_n) \in S$
- Strategy profile without player i: $s_{-i} = (s_1, s_2, ..., s_{i-1}, s_{i+1}, \ldots, s_n)$
- $u_i : S \to \mathbb{R}$, utility function of player i

- **Normal form** representation is a tuple $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$
- If S_i is finite $\forall i \in N$, this is called a finite game.

Example: Penalty Shoot Game

		Goalkeeper		
		L	C	R
Shooter	L	$-1, 1$	$1, -1$	$1, -1$
	C	$1, -1$	$-1, 1$	$1, -1$
	R	$1, -1$	$1, -1$	$-1, 1$

- $N = \{1, 2\}, 1 = \text{Shooter}, 2 = \text{Goalkeeper}$
- $S_1 = S_2 = \{L, C, R\}$
- $u_1(L, L) = -1, u_1(L, C) = u_1(L, R) = 1$
- $u_2(L, L) = 1, u_2(L, C) = u_2(L, R) = -1$
- (loosely) $u_1(X, X) = -1 = -u_2(X, X), u_1(X, Y) = -u_2(X, Y) = 1$

Contents

Domination in NFGs

		Player 2		
		L	C	R
Player 1	U	1,0	1,3	3,2
	D	−1,6	0,5	3,3

Question

Will a **rational** Player 2 ever play R?

Dominated Strategy

Definition (Strictly Dominated Strategy)

A strategy $s_i' \in S_i$ of player i is **strictly dominated** if there exists another strategy $s_i \in S_i$ such that **for every strategy profile** $s_{-i} \in S_{-i}$ of the other players, $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$.

Definition (Weakly Dominated Strategy)

A strategy $s_i' \in S_i$ of player i is **weakly dominated** if there exists another strategy $s_i \in S_i$ such that **for every strategy profile** $s_{-i} \in S_{-i}$ of the other players $u_i(s_i, s_{-i}) \geqslant u_i(s_i', s_{-i})$ and **there exists some** $\tilde{s}_{-i} \in S_{-i}$ such that $u_i(s_i, \tilde{s}_{-i}) > u_i(s_i', \tilde{s}_{-i})$.

Example: R is strictly dominated (by C) while D is weakly dominated (by U)

Dominant Strategy

A strategy s'_i can be dominated by s_i, and a different strategy s''_i can be dominated by $\tilde{s}_i$

Definition (Dominant Strategy)

A strategy s_i is strictly(weakly) dominant strategy for player i if s_i strictly(weakly) dominates all other strategies $s'_i \in S_i \setminus \{s_i\}$.

Examples of **dominant strategy**

- Neighbouring kingdom's dilemma
- Indivisible item for sale

Neighbouring Kingdom’s Dilemma

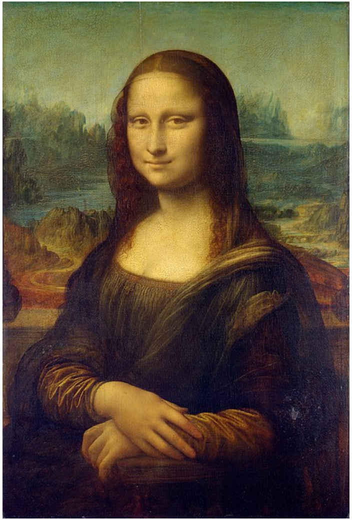
		Rashtrakuta	
		Agri	War
Pala	Agri	5,5	0,6
	War	6,0	1,1

Question

Is there a dominant strategy in this game? Which kind?

Indivisible Item for Sale

- Two players value an indivisible item as v_1 and v_2 respectively
- Each player's action: a number in $[0, M]$, $M \gg v_1, v_2$
- Player quoting the larger number wins the object (ties broken in favour of player 1) and pays the losing player's chosen number
- utility of winning player = her **true** value - other player's chosen number
- utility of losing player = 0



Indivisible Item for Sale

Normal form representation of the game

- $N = \{1, 2\}$, $S_1 = S_2 = [0, M]$
- Agents pick s_i, while their **real** value for the item is v_i, and s_i may **not** be the same as v_i

$$u_1(s_1, s_2) = \begin{cases} v_1 - s_2 & \text{if } s_1 \geqslant s_2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$u_2(s_1, s_2) = \begin{cases} v_2 - s_1 & \text{if } s_1 < s_2 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Question

Is there a dominant strategy in this game? Which kind?

Dominant Strategy Equilibrium

Definition (Dominant Strategy Equilibrium)

A strategy profile $(s_1^*, s_2^*, \ldots, s_n^*)$ is a strictly (weakly) dominant strategy equilibrium (SDSE/WDSE) if s_i^* is strictly (weakly) dominant strategy $\forall i \in N$.

Example of **dominant strategy equilibrium**

		Player 2	
		D	E
Player 1	A	5,5	0,5
	B	5,0	1,1
	C	4,0	1,1

Question

What kind of equilibrium in this game?

How to find equilibrium?

- Rational players do not play **dominated strategies**
- To obtain rational outcomes eliminate dominated strategies
- For strictly dominated strategies the order of elimination does not matter
- It matters for weakly dominated strategies – some reasonable outcomes are also eliminated

Player 2

		L	C	R
Player 1	T	1,2	2,3	0,3
	M	2,2	2,1	3,2
	B	2,1	0,0	1,0

- Order T, R, B, C $\rightarrow (M, L) : (2, 2)$
- Order B, L, C, T $\rightarrow (M, R) : (3, 2)$

Existence of Dominant Strategies

Not guaranteed!

		Player 2	
		L	R
Player 1	L	1, 1	0, 0
	R	0, 0	1, 1

Co-ordination game

		Friend 2	
		F	C
Friend 1	F	2, 1	0, 0
	C	0, 0	1, 2

Football or Cricket Game

If **dominance** cannot explain a reasonable outcome – refine equilibrium concept

Contents

Nash Equilibrium (Nash 1951)

No player gains by a unilateral deviation

Definition (Nash Equilibrium)

A strategy profile (s_i^*, s_{-i}^*) is a pure strategy Nash equilibrium (PSNE) if $\forall i \in N$ and $\forall s_i \in S_i$

$$u_i(s_i^*, s_{-i}^*) \geqslant u_i(s_i, s_{-i}^*).$$

		Friend 2	
		F	C
Friend 1	F	2,1	0,0
	C	0,0	1,2

Football or Cricket Game

Best Response View

- A best response of a player i against the strategy profile s_{-i} of other players is a strategy that gives the maximum utility i.e.,

$$B_i(s_{-i}) = \{s_i \in S_i : u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}), \forall s_i' \in S_i\}$$

- PSNE is a strategy profile (s_i^*, s_{-i}^*) s.t.

$$s_i^* \in B_i(s_{-i}^*), \forall i \in N$$

- PSNE gives stability – once there, no rational player unilaterally deviates

Question

Relationship between SDSE, WDSE, PSNE?

Answer

SDSE $\implies$ WDSE $\implies$ PSNE

Contents

Risk Aversion of Players

Player 2

		L	R
Player 1	T	$2, 1$	$1, -20$
	M	$3, 0$	$-10, 1$
	B	$-100, 2$	$3, 3$

Question

What if the other player does not pick an equilibrium action (Nash)?

Picking T is less risky for player 1

Max-min Strategy

Definition

The worst case optimal choice is **max-min strategy**

$$s_i^{\max\min} \in \arg\max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$$

Note: $s_{-i}^{\min}(s_i) \in \arg\min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$ is indeed a function of s_i; as s_i changes the minimizer keeps on changing

Max-min value (utility at the max-min strategy) of player i is given by

$$\underline{v}_i = \max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$$

$$u_i(s_i^{\max\min}, t_{-i}) \geqslant \underline{v}_i, \qquad \forall t_{-i} \in S_{-i}$$

Max-min and Dominant Strategies

Theorem

*If s_i^* is **dominant strategy** for player i, then it is a **max-min strategy** for player i as well.*

Proof.

Let s_i^* be dominant strategy for player i

$$u_i(s_i^*, s_{-i}) \geqslant u_i(s_i', s_{-i}),\ \forall s_{-i} \in S_{-i}, \forall s_i' \in S_i \setminus \{s_i^*\} \tag{3}$$

Define $s_{-i}^{\min}(s_i') \in \arg\min_{s_{-i} \in S_{-i}} u_i(s_i', s_{-i})$: the worst choice of strategies of the other players for the action s_i' of agent i

But Equation (3) holds for every s_{-i}, in particular $s_{-i}^{\min}(s_i')$

$$u_i(s_i^*, s_{-i}^{\min}(s_i')) \geqslant u_i(s_i', s_{-i}^{\min}(s_i')),\ \forall s_i' \in S_i \setminus \{s_i^*\}$$

$$s_i^* \in \arg\max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$$

Max-min and PSNE

Theorem

Every ***PSNE*** $s^* = (s_1^*, s_2^*, \dots, s_n^*)$ *of a* normal form game *satisfies* $u_i(s^*) \geqslant \underline{v}_i,\ \forall i \in N$.

Proof.

$$
\begin{aligned}
u_i(s_i, s_{-i}^*) &\geqslant \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}),\ \forall s_i \in S_i, && \text{by definition of min} \\
u_i(s_i^*, s_{-i}^*) &\geqslant u_i(s_i, s_{-i}^*),\ \forall s_i \in S_i, && \text{by definition of PSNE} \\
u_i(s_i^*, s_{-i}^*) &= \max_{s_i \in S_i} u_i(s_i, s_{-i}^*) \geqslant \max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) = \underline{v}_i
\end{aligned}
$$

□

Contents

Iterated elimination of dominated strategies

The story so far

- Dominance cannot explain all outcomes; games may not have dominant strategies
- PSNE: unilateral deviation; gives **stability**
- Maxmin: rationality for risk-aversion; gives **security**

Question

What happens to stability and security when some strategies are eliminated?

Iterated elimination of dominated strategies (contd.)

		Player 2		
		L	C	R
Player 1	T	1, 2	2, 3	0, 3
	M	2, 2	2, 1	3, 2
	B	2, 1	0, 0	1, 0

- Order T, R, B, C → $(M, L) : (2, 2)$
- Order B, L, C, T → $(M, R) : (3, 2)$

Question

Does it change the maxmin value?

Iterated elimination of dominated strategies (contd.)

Player 2

		L	C	R
Player 1	T	1,2	2,3	0,3
	M	2,2	2,1	3,2
	B	2,0	0,0	1,0

Consider in the above example: elimination of dominated strategy B for player 1

Maxmin values	Player 1	Player 2
Before	2	0
After	2	2

Maxmin value is not affected for the player whose **dominated strategy** is removed

A Result for Iterated Elimination

Theorem

Consider an NFG $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$*, and let* $s_j' \in S_j$ *be a dominated strategy. Let* G' *be the residual game after removing* s_j'*. Then, the maxmin value of* j *in* G' *is equal to her maxmin value in* G*.*

Intuition

- Maxmin is the ‘max’ of all ‘min’s
- Elimination affects one ‘min’
- But that does not affect the ‘max’ since the strategy was dominated

Proof

Maxmin value of player j in G $$\underline{\mathrm{v}}_j = \max_{s_j \in S_j} \min_{s_{-j} \in S_{-j}} u_j(s_j, s_{-j})$$

Maxmin value of player j in G' $$\underline{\mathrm{v}}'_j = \max_{s_j \in S_j \setminus \{s'_j\}} \min_{s_{-j} \in S_{-j}} u_j(s_j, s_{-j})$$

Suppose t_j dominates s'_j in $G, t_j \in S_j \setminus \{s'_j\},$ then, $$u_j(t_j, s_{-j}) \geqslant u_j(s'_j, s_{-j}), \forall s_{-j} \in S_{-j}$$

Therefore, $$\min_{s_{-j} \in S_{-j}} u_j(t_j, s_{-j}) = u_j(t_j, \tilde{s}_{-j}) \geqslant u_j(s'_j, \tilde{s}_{-j}) \geqslant \min_{s_{-j} \in S_{-j}} u_j(s'_j, s_{-j})$$

$$\implies \max_{s_j \in S_j \setminus \{s'_j\}} \min_{s_{-j} \in S_{-j}} u_j(s_j, s_{-j}) \geqslant \min_{s_{-j} \in S_{-j}} u_j(t_j, s_{-j}) \geqslant \min_{s_{-j} \in S_{-j}} u_j(s'_j, s_{-j})$$

Proof (contd.)

$$
\begin{aligned}
&\underline{\mathrm{v}}_j \quad \text{[maxmin value of } j \text{ in } G\text{]} \\
&= \max_{s_j \in S_j} \min_{s_{-j} \in S_{-j}} u_j(s_j, s_{-j}) \\
&= \max \left\{ \max_{s_j \in S_j \setminus \{s_j'\}} \min_{s_{-j} \in S_{-j}} u_j(s_j, s_{-j}), \min_{s_{-j} \in S_{-j}} u_j(s_j', s_{-j}) \right\} \\
&= \max_{s_j \in S_j \setminus \{s_j'\}} \min_{s_{-j} \in S_{-j}} u_j(s_j, s_{-j}), \text{ because of the previous inequality} \\
&= \underline{\mathrm{v}}_j' \quad \text{[maxmin value of j in } G'\text{]}
\end{aligned}
$$

Contents

Preservation of PSNE

Question

What happens to existing equilibrium after iterated elimination?

Theorem

Consider G and $\hat{G}$ are games before and after elimination of a strategy (not necessarily dominated). If s^* is a PSNE in G and survives in $\hat{G}$, then s^* is a PSNE in $\hat{G}$ too.

Intuition

PSNE was a maxima of utility of i among the strategies of i. Removing other strategies does not affect maximality.
Proof: exercise.

Can new equilibrium be generated?

Theorem

Consider NFG G. Let $\hat{s}_j$ be a weakly dominated strategy of j. If $\hat{G}$ is obtained from G eliminating $\hat{s}_j$, then every PSNE of $\hat{G}$ is a PSNE of G.

No new PSNE if the eliminated strategy is dominated

But old PSNEs could be killed: saw in the previous example

Proof

In the game $\hat{G}$, modified strategy sets are $\hat{S}_j = S_j \setminus \{\hat{s}_j\}$, $\hat{S}_i = S_i, \forall i \neq j$

Need to show: if $s^* = (s_j^*, s_{-j}^*)$ is a PSNE in $\hat{G}$, it is a PSNE in G.

Given

$$u_i(s^*) \geqslant u_i(s_i, s_{-i}^*), \forall i \neq j, \forall s_i \in \hat{S}_i = S_i$$

$$u_j(s^*) \geqslant u_j(s_j, s_{-j}^*), \forall s_j \in \hat{S}_j$$

Need to show: no profitable deviation for any player in G. For $i \neq j$, this is immediate since no strategy is removed.

For j, no profitable deviation from s^* for any strategy $s_j \neq \hat{s}_j$
Since $\hat{s}_j$ is dominated, $\exists t_j$ such that

$$u_j(t_j, s_{-j}) \geqslant u_j(\hat{s}_j, s_{-j}), \forall s_{-j} \in S_{-j}$$

In particular, $u_j(t_j, s_{-j}^*) \geqslant u_j(\hat{s}_j, s_{-j}^*)$

Since s^* is a PSNE in $\hat{G}$ and $t_j \in \hat{S}_j$

$$u_j(s_j^*, s_{-j}^*) \geqslant u_j(t_j, s_{-j}^*) \geqslant u_j(\hat{s}_j, s_{-j}^*)$$

Summary

- Elimination of strictly dominated strategy have no effect on PSNE
- Elimination of weakly dominated strategy may reduce the set of PSNEs, but never adds new
- The maxmin values of the player whose strictly or weakly dominated strategies are remove remain unaffected

Contents

Matrix games: *two player zero-sum* games

A special class with certain nice **security** and **stability** properties

Definition (Two player zero-sum games)

A NFG $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ with $N = \{1, 2\}$ and $u_1 + u_2 \equiv 0$

Question

Why called **matrix** game?

Answer

Possible to represent the game with only one matrix considering the utilities of player 1; player 2's utilities are negative of this matrix

Example: Penalty shoot game

		Player 2	
		L	R
Player 1	L	$-1, 1$	$1, -1$
	R	$1, -1$	$-1, 1$

$$\Longrightarrow \qquad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} =: u$$

Player 2's ***maxmin*** *value is the* ***minmax*** *value of this matrix*

		L	*R*	*maxmin*
Player 1	*L*	-1	1	-1
	R	1	-1	-1
	minmax	1	1	

भारतीय प्रौद्योगिकी संस्थान मुंबई

Indian Institute of Technology Bombay