

भारतीय प्रौद्योगिकी संस्थान मुंबई

Indian Institute of Technology Bombay

CS 6001: Game Theory and Algorithmic Mechanism Design

Week 3

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Slide preparation acknowledgments: Onkar Borade and Rounak Dalmia

ज्ञानम् परमम् ध्येयम् Knowledge is the supreme goal

Contents



- ► Matrix games
- ► Relation between **maxmin** and PSNE
- ► Mixed Strategies
- ► Mixed Strategy Nash Equilibrium
- ► Find MSNE
- ► MSNE Characterization Theorem Proof
- ► Algorithm to find MSNE
- ► Existence of MSNE



A special class with certain nice **security** and **stability** properties



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Definition (Two player zero-sum games)

A NFG
$$\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$$
 with $N = \{1, 2\}$ and $u_1 + u_2 \equiv 0$



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Question

Why called **matrix** game?



A special class with certain nice security and stability properties

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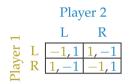
Why called matrix game?

Answer

Possible to represent the game with only one matrix considering the utilities of player 1; player 2's utilities are negative of this matrix

Example: Penalty shoot game





Example: Penalty shoot game



Player 2

L R

L
$$-1,1$$
 $1,-1$

R $1,-1$ $-1,1$

$$\Longrightarrow$$

$$\left(\begin{array}{cc} -1 & 1 \\ 1 & -1 \end{array}\right) =: u$$

L	R maxmin		
-1	1	-1	
1	-1	-1	
1	1		

Example: Penalty shoot game



Player 2
$$\begin{array}{ccc}
L & R \\
\hline
 & L & -1, 1 & 1, -1 \\
\hline
 & R & 1, -1 & -1, 1
\end{array}$$

$$\Longrightarrow \qquad \left(\begin{array}{ccc}
-1 & 1 \\
1 & -1
\end{array}\right) =: u$$

Player 2's maxmin value is the minmax value of this matrix

		L	IX I	пахиш
7	L	-1	1	-1
aye	R	1	-1	-1
Pl	minmax	1	1	

Another example



Player 2 L C R T 3,-3 -5,5 -2,2 M 1,-1 4,-4 1,-1 B 6,-6 -3,3 -5,5

Another example



		I	Player !	2			L	C	R r	naxmir	ı
		L	C	R		T	3	-5	-2	-5	
	T	3, -3	-5,5	-2,2	er 1	M	1	4	1	1	
ayer	M	1, -1	$\frac{4}{4}$, -4	<mark>1, −1</mark>	Play	В	6	-3	-5	-5	
Pl	В	6, -6	-3,3	-5,5		minmax	6	4	1		

Two examples together



R

minmax

R maxmin

minmax

M

L	C	R r	naxmir
3	-5	-2	-5
1	4	1	1
6	-3	-5	-5
6	4	1	

Two examples together



ľ 1	L	-1	
aye	R	1	
Pl	minmax	1	

R maxmin		
1	-1	
-1	-1	
1		
	R r 1 -1 1	

	1
er 1	M
Play	В
	minmax

L	C	R maxm	
3	-5	-2	-5
1	4	1	1
6	-3	-5	-5
6	4	1	

Question

What are the PSNEs for the above games?

Two examples together



	L
ауе	R
Pl	minma

	L	R r	naxmii
	-1	1	-1
	1	-1	-1
ax	1	1	

Player 1

B minmax

L	C	R r	naxmin
3	-5	-2	-5
1	4	1	1
6	-3	-5	-5
6	4	1	

Question

What are the PSNEs for the above games?

Answer

Left: no PSNE; Right: (M,R)



Saddle point of a matrix

The value is simultaneously the maximum in its column and minimum in its row i.e., maximum for player 1 and minimum for player 2



Saddle point of a matrix

The value is simultaneously the maximum in its column and minimum in its row i.e., maximum for player 1 and minimum for player 2

Question

What are the saddle points for the previous two games?



_		L	R
er	L	-1	1
Play	R	1	-1

	L	C	R
- T	3	-5	-2
M del	1	4	1
В	6	-3	-5



	L	R
P F	-1	1
lay R	1	-1

	L	C	R
T	3	-5	-2
M	1	4	1
В	6	-3	-5

Answer

For the first example: no saddle point, for the second: (M,R)



_		L	R
er)	L	-1	1
Play	R	1	-1

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ayeı M	1	4	1
ı B	6	-3	-5

Answer

For the first example: no saddle point, for the second: (M,R)

Theorem

In a matrix game with utility matrix u, (s_1^*, s_2^*) is a saddle point iff it is a PSNE.

Saddle point and PSNE



Proof.

Consider (s_1^*, s_2^*) to be a saddle point. By definition of saddle point, this happens iff $u(s_1^*, s_2^*) \geqslant u(s_1, s_2^*), \forall s_1 \in S_1$ and $u(s_1^*, s_2^*) \leqslant u(s_1^*, s_2), \forall s_2 \in S_2$. Since, $u \equiv u_1 \equiv -u_2$, the above is equivalent to $u_1(s_1^*, s_2^*) \geqslant u_1(s_1, s_2^*), \forall s_1 \in S_1$ and $u_2(s_1^*, s_2^*) \geqslant u_2(s_1^*, s_2), \forall s_2 \in S_2 \Leftrightarrow (s_1^*, s_2^*)$ is a PSNE.

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Consider **maxmin** and **minmax** values

$$\underline{v} = \max_{s_1 \in S_1} \min_{s_2 \in S_2} u(s_1, s_2)$$

$$\overline{v} = \min_{s_2 \in S_2} \max_{s_1 \in S_1} u(s_1, s_2)$$

minmax

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$$\underline{v} = \max_{s_1 \in S_1} \min_{s_2 \in S_2} u(s_1, s_2)$$
 maxmin

$$\overline{v} = \min_{s_2 \in S_2} \max_{s_1 \in S_1} u(s_1, s_2)$$
 minmax

Question

How are the maxmin and minmax values related?



Lemma

For matrix games $\overline{v} \geqslant \underline{v}$.



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Proof.

$$u(s_1, s_2) \geqslant \min_{t_2 \in S_2} u(s_1, t_2), \ \forall s_1, s_2,$$

definition of min



Lemma

For matrix games $\overline{v} \geqslant \underline{v}$.

Proof.

$$u(s_1, s_2) \geqslant \min_{t_2 \in S_2} u(s_1, t_2), \ \forall s_1, s_2,$$

 $\Rightarrow \max_{t_1 \in S_1} u(t_1, s_2) \geqslant \max_{t_1 \in S_1} \min_{t_2 \in S_2} u(t_1, t_2), \ \forall s_2 \in S_2$

definition of min

RHS was dominated for each s_1



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For matrix games $\overline{v} \geqslant \underline{v}$.

Proof.

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$$\Rightarrow \min_{t_{2} \in S_{2}} \max_{t_{1} \in S_{1}} u(t_{1}, t_{2}) \geqslant \max_{t_{1} \in S_{1}} \min_{t_{2} \in S_{2}} u(t_{1}, t_{2})$$

definition of min

RHS was dominated for each s_1 RHS was a constant

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Earlier examples and security values



		L	R	maxmin
Player 1	L	-1	1	-1
	R	1	-1	-1
	minmax	1	1	

Earlier examples and security values



		L	R	maxmin
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Player 1	R	1	-1	-1
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$$\overline{v} = 1 > -1 = \underline{v}$$

Earlier examples and security values



		L	R	maxmin
Player 1	L	-1	1	-1
	R	1	-1	-1
Ξ.	minmax	1	1	

 $\overline{v} = 1 > -1 = \underline{v}$ PSNE does not exist

Earlier examples and security values (contd.)



		L	C	R	maxmin
Player 1	T	3	-5	-2	-5
	M	1	4	1	1
	В	6	-3	-5	-5
	minmax	6	4	1	

Earlier examples and security values (contd.)



\vdash	
Player	

M

В

minma

	L	С	R	maxmin
	3	-5	-2	-5
	1	4	1	1
	6	-3	-5	-5
ax	6	4	1	

$$\overline{v} = 1 = \underline{v}$$

Earlier examples and security values (contd.)



		L	C	K	maxmin
Player 1 W B	T	3	-5	-2	-5
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	minmax	6	4	1	

$$\overline{v} = 1 = \underline{v}$$

PSNE exists

PSNE Theorem



Define the following strategies

$$s_1^* \in \arg \max_{s_1 \in S_1} \min_{s_2 \in S_2} u(s_1, s_2),$$

$$s_2^* \in \arg\min_{s_2 \in S_2} \max_{s_1 \in S_1} u(s_1, s_2),$$

maxmin strategy of player 1

minmax strategy of player 2

PSNE Theorem



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maxmin strategy of player 1

minmax strategy of player 2

Theorem

A game has a PSNE (equivalently, a saddle point) if and only if $\overline{v} = \underline{v} = u(s_1^*, s_2^*)$, where s_1^* and s_2^* are maxmin and minmax strategies for players 1 and 2 respectively.

Corollary: (s_1^*, s_2^*) is a PSNE

Proof of the PSNE Theorem



Proof

(\Longrightarrow) let (s_1^*,s_2^*) is a PSNE $\Longrightarrow \overline{v}=\underline{v}=u(s_1^*,s_2^*)$ and s_1^* and s_2^* are maxmin and minmax



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$$\implies u(s_1^*, s_2^*) \geqslant \max_{t_1 \in S_1} u(t_1, s_2^*)$$

$$\geqslant \min_{t_2 \in S_2} \max_{t_1 \in S_1} u(t_1, t_2), \text{ since } s_2^* \text{ is a specific strategy}$$

$$= \overline{v}$$



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$$u(s_1^*, s_2^*) \geqslant \overline{v} \geqslant \underline{v} \geqslant u(s_1^*, s_2^*)$$

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$$\implies u(s_1^*, s_2^*) = \overline{v} = \underline{v}$$

Also implies that the maxmin for 1 and minmax for 2 are s_1^* and s_2^* respectively.



Proof (contd.)

(
$$\leftarrow$$
) i.e. $\overline{v} = \underline{v} = u(s_1^*, s_2^*)$ and s_1^* and s_2^* are maxmin and minmax $\implies (s_1^*, s_2^*)$ is a PSNE



Proof (contd.)

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$$u(s_1^*, s_2) \geqslant \min_{t_2 \in S_2} u(s_1^*, t_2)$$
, by definition of min
$$= \max_{t_1 \in S_1} \min_{t_2 \in S_2} u(t_1, t_2)$$
, since s_1^* is the maxmin strategy for player 1
$$= v \text{ (given)}$$



Proof (contd.)

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) i.e. $\overline{v} = \underline{v} = u(s_1^*, s_2^*)$ and s_1^* and s_2^* are maxmin and minmax $\implies (s_1^*, s_2^*)$ is a PSNE

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Similarly, we can show $u(s_1, s_2^*) \leq v$, $\forall s_1 \in S_1$



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$$=\max_{t_1\in S_1}\min_{t_2\in S_2}u(t_1,t_2), \text{ since } s_1^* \text{ is the maxmin strategy for player 1}$$

$$=v \text{ (given)}$$

Similarly, we can show $u(s_1, s_2^*) \leq v$, $\forall s_1 \in S_1$ But $v = u(s_1^*, s_2^*)$. Substitute and get that (s_1^*, s_2^*) is a PSNE.

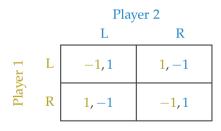
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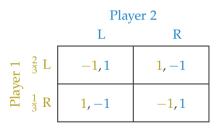


Mixed strategy: probability distribution over the set of strategies of that player



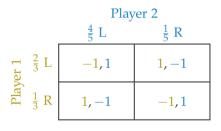


Mixed strategy: probability distribution over the set of strategies of that player





Mixed strategy: probability distribution over the set of strategies of that player





Mixed strategy: probability distribution over the set of strategies of that player

Player 2
$$\frac{4}{5}$$
 L $\frac{1}{5}$ R

1.1 $\frac{2}{3}$ L $\frac{2}{3}$ L $\frac{1}{5}$ R

1.1 1, -1 $\frac{1}{5}$ R

1.1 1, -1 $\frac{1}{5}$ R

• Consider a finite set *A*, define

$$\Delta A = \{ p \in [0,1]^{|A|} : \sum_{a \in A} p(a) = 1 \}$$



Mixed strategy: probability distribution over the set of strategies of that player

Player 2
$$\frac{4}{5}$$
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• Consider a finite set *A*, define

$$\Delta A = \{ p \in [0,1]^{|A|} : \sum_{a \in A} p(a) = 1 \}$$

• Mixed strategy set of player 1: $\Delta S_1 = \Delta \{L, R\}, (\frac{2}{3}, \frac{1}{3}) \in \Delta S_1$



• **Notation**: σ_i is a mixed strategy of player i



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- $\sigma_i \in \Delta S_i$, i.e., $\sigma_i : S_i \to [0,1]$ s.t. $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$



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- We are discussing non-cooperative games, the players choose their strategies independently



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- The joint probability of player 1 picking s_1 and player 2 picking $s_2 = \sigma_1(s_1)\sigma_2(s_2)$



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- The joint probability of player 1 picking s_1 and player 2 picking $s_2 = \sigma_1(s_1)\sigma_2(s_2)$
- Utility of player i at a mixed strategy profile (σ_i, σ_{-i}) is

$$u_i(\sigma_i,\sigma_{-i}) = \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} \cdots \sum_{s_n \in S_n} \sigma_1(s_1) \cdot \sigma_2(s_2) \cdots \sigma_n(s_n) \ u_i(s_1,s_2,\ldots,s_n)$$



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- Utility of player *i* at a mixed strategy profile (σ_i, σ_{-i}) is

$$u_i(\sigma_i,\sigma_{-i}) = \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} \cdots \sum_{s_n \in S_n} \sigma_1(s_1) \cdot \sigma_2(s_2) \cdots \sigma_n(s_n) \ u_i(s_1,s_2,\ldots,s_n)$$

• We are overloading u_i to denote the utility at pure and mixed strategies



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- $\sigma_i \in \Delta S_i$, i.e., $\sigma_i : S_i \to [0,1]$ s.t. $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$
- We are discussing non-cooperative games, the players choose their strategies independently
- The joint probability of player 1 picking s_1 and player 2 picking $s_2 = \sigma_1(s_1)\sigma_2(s_2)$
- Utility of player *i* at a mixed strategy profile (σ_i, σ_{-i}) is

$$u_i(\sigma_i,\sigma_{-i}) = \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} \cdots \sum_{s_n \in S_n} \sigma_1(s_1) \cdot \sigma_2(s_2) \cdots \sigma_n(s_n) \ u_i(s_1,s_2,\ldots,s_n)$$

- We are overloading u_i to denote the utility at pure and mixed strategies
- Utility at a mixed strategy is the **expectation** of the utilities at pure strategies; all the rules of expectation hold, e.g., linearity, conditional expectation, etc.



			Player 2	
			$\frac{4}{5}$ L	$\frac{1}{5}$ R
Player 1	$\frac{2}{3}$ I	.	-1,1	1 , −1
	$\frac{1}{3}$ F	2	1, -1	-1,1



Player 2
$$\frac{4}{5}$$
 L $\frac{1}{5}$ R

1 $\frac{2}{3}$ L $-1,1$ 1,-1
 $\frac{1}{3}$ R 1,-1 $-1,1$

$$u_1(\sigma_1,\sigma_2)$$



Player 2
$$\frac{4}{5}$$
 L $\frac{1}{5}$ R

1 $\frac{2}{3}$ L $\frac{2}{3}$ L $\frac{1}{5}$ R

1,-1 $\frac{1}{5}$ R

$$u_1(\sigma_1, \sigma_2) = \frac{2}{3} \cdot \frac{4}{5} \cdot (-1)$$



$$u_1(\sigma_1, \sigma_2) = \frac{2}{3} \cdot \frac{4}{5} \cdot (-1) + \frac{2}{3} \cdot \frac{1}{5} \cdot (1)$$



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$$\frac{4}{5}$$
 L $\frac{1}{5}$ R

1 $\frac{2}{3}$ L $\frac{2}{3}$ R $\frac{1}{3}$ R $\frac{1}$

$$u_1(\sigma_1, \sigma_2) = \frac{2}{3} \cdot \frac{4}{5} \cdot (-1) + \frac{2}{3} \cdot \frac{1}{5} \cdot (1) + \frac{1}{3} \cdot \frac{4}{5} \cdot (1)$$



Player 2
$$\frac{4}{5}$$
 L $\frac{1}{5}$ R

1 $\frac{2}{3}$ L $-1,1$ 1,-1
 $\frac{1}{3}$ R 1,-1 $-1,1$

$$u_1(\sigma_1, \sigma_2) = \frac{2}{3} \cdot \frac{4}{5} \cdot (-1) + \frac{2}{3} \cdot \frac{1}{5} \cdot (1) + \frac{1}{3} \cdot \frac{4}{5} \cdot (1) + \frac{1}{3} \cdot \frac{1}{5} \cdot (-1)$$

Contents



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- ► Relation between **maxmin** and PSNE
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- ► Existence of MSNE

Mixed Strategies Nash Equilibrium



Definition (Mixed Strategy Nash Equilibrium)

A mixed strategy Nash equilibrium (MSNE) is a mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$, s.t.

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geqslant u_i(\sigma_i, \sigma_{-i}^*), \ \forall \sigma_i \in \Delta S_i \text{ and } \forall i \in N.$$

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Question

Relation between **PSNE** and **MSNE**?

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Question

Relation between **PSNE** and **MSNE**?

Answer

 $PSNE \implies MSNE$

An Alternative Definition



Theorem

A mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$, is an **MSNE** if and only if $\forall s_i \in S_i$ and $\forall i \in N$

$$u_i(\sigma_i^*,\sigma_{-i}^*) \geq u_i(s_i,\sigma_{-i}^*).$$

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Proof.

 (\Rightarrow) : The pure strategy s_i is a special case of the mixed strategy, the mixed strategy with s_i having probability 1. Inequality holds by definition of MSNE

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 (\Rightarrow) : The pure strategy s_i is a special case of the mixed strategy, the mixed strategy with s_i having probability 1. Inequality holds by definition of MSNE (\Leftarrow) Pick an arbitrary mixed strategy σ_i of player i

$$u_i(\sigma_i, \sigma_{-i}^*) = \sum_{s_i \in S_i} \sigma_i(s_i) \cdot u_i(s_i, \sigma_{-i}^*)$$

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$$\begin{aligned} u_i(\sigma_i, \sigma_{-i}^*) &= \sum_{s_i \in S_i} \sigma_i(s_i) \cdot u_i(s_i, \sigma_{-i}^*) \\ &\leqslant \sum_{s_i \in S_i} \sigma_i(s_i) \cdot u_i(\sigma_i^*, \sigma_{-i}^*) \end{aligned}$$

3

An Alternative Definition



Theorem

A mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$, is an **MSNE** if and only if $\forall s_i \in S_i$ and $\forall i \in N$

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Proof.

(⇒): The pure strategy s_i is a special case of the mixed strategy, the mixed strategy with s_i having probability 1. Inequality holds by definition of MSNE (\Leftarrow) Pick an arbitrary mixed strategy σ_i of player i

$$u_{i}(\sigma_{i}, \sigma_{-i}^{*}) = \sum_{s_{i} \in S_{i}} \sigma_{i}(s_{i}) \cdot u_{i}(s_{i}, \sigma_{-i}^{*})$$

$$\leq \sum_{s_{i} \in S_{i}} \sigma_{i}(s_{i}) \cdot u_{i}(\sigma_{i}^{*}, \sigma_{-i}^{*})$$

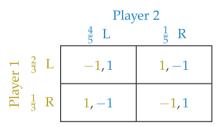
$$= u_{i}(\sigma_{i}^{*}, \sigma_{-i}^{*}) \cdot \sum_{s_{i} \in S_{i}} \sigma_{i}(s_{i}) = u_{i}(\sigma_{i}^{*}, \sigma_{-i}^{*})$$

3



Question

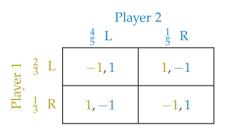
Is the mixed strategy profile an **MSNE**?



• To answer this, we need to show that there does not exist any better mixed strategy for the player



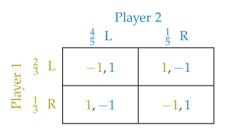
Question



- To answer this, we need to show that there does not exist any better mixed strategy for the player
- Expected utility of player 2 from $L = 2/3 \cdot 1 + 1/3 \cdot (-1) = 1/3$



Question



- To answer this, we need to show that there does not exist any better mixed strategy for the player
- Expected utility of player 2 from $L = 2/3 \cdot 1 + 1/3 \cdot (-1) = 1/3$
- Expected utility of player 2 from $R = 2/3 \cdot (-1) + 1/3 \cdot 1 = -1/3$



Question

Is the mixed strategy profile an **MSNE**?

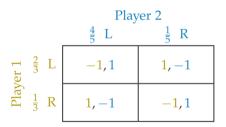
Player 2
$$\frac{4}{5}$$
 L $\frac{1}{5}$ R

1 Jack 1 1, -1 1, -1
 $\frac{2}{3}$ R 1, -1 -1, 1

Expected utility will increase if some probability is transferred from R to L



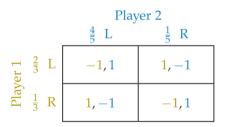
Question



- Expected utility will increase if some probability is transferred from R to L
- ⇒ the current profile is **not** an MSNE



Question



- Expected utility will increase if some probability is transferred from R to L
- \Rightarrow the current profile is **not** an MSNE
- Some balance in the utilities is needed



Question

		Player 2		
		$\frac{1}{2}$ L	$\frac{1}{2}$ R	
51	$\frac{1}{2}$ L	-1,1	1, -1	
Playe	$\frac{1}{2}$ R	1, -1	-1,1	

- Expected utility will increase if some probability is transferred from R to L
- \Rightarrow the current profile is **not** an MSNE
- Some balance in the utilities is needed
- Does there exist any improving mixed strategy?

Contents



- ► Matrix games
- ▶ Relation between **maxmin** and PSNE
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- ► Find MSNE
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Definition (Support of mixed strategy/probability distribution)

For mixed strategy σ_i , the subset of strategy set of i on which σ_i has a positive mass is called the **support** of σ_i and is denoted by $\delta(\sigma_i)$. Formally, $\delta(\sigma_i) = \{s_i \in S_i : \sigma_i(s_i) > 0\}$.



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Using the definition of support, here is a characterization of MSNE

Theorem

A mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ is an MSNE iff $a \forall i \in N$

^aThis is a shorthand for 'if and only if'.



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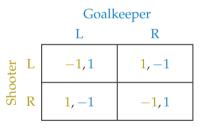


Consider Penalty Shoot Game

		Goalkeeper		
		L	R	
Shooter	L	-1,1	1, -1	
	R	1, -1	-1,1	



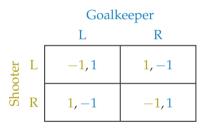
Consider Penalty Shoot Game



Case 1: support profile $(\{L\}, \{L\})$: for player 1, $s'_1 = R$ – violates condition 2



Consider Penalty Shoot Game



Case 1: support profile $(\{L\}, \{L\})$: for player 1, $s'_1 = R$ – violates condition 2

Case 2: support profile $(\{L, R\}, \{L\})$ – symmetric for the other case

For Player 1, the expected utility has to be the same for L and R - not possible – violates condition 1



Case 3: support profile $(\{L, R\}, \{L, R\})$: condition 2 is vacuously satisfied



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For condition 1, let player 1 chooses L w.p. p and player 2 choose L w.p. q



Case 3: support profile ($\{L,R\}$, $\{L,R\}$): condition 2 is vacuously satisfied

For condition 1, let player 1 chooses L w.p. p and player 2 choose L w.p. q

For player 1:

$$u_1(L,(q,1-q)) = u_1(R,(q,1-q)) \Rightarrow (-1)q + 1 \cdot (1-q) = 1 \cdot q + (-1)(1-q) \Rightarrow q = \frac{1}{2}$$



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For player 2:

$$u_2((p,1-p),L) = u_2((p,1-p),R) \Rightarrow p = \frac{1}{2}$$



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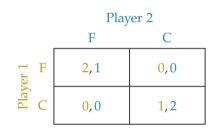
For player 2:

$$u_2((p,1-p),L) = u_2((p,1-p),R) \Rightarrow p = \frac{1}{2}$$

$$\left(\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{2}\right)\right)$$

Exercises





	Player 2				
	F	С	D		
er 1	2,1	0,0	1,1		
Play O	0,0	1,2	2,0		

Contents



- ► Matrix games
- ▶ Relation between **maxmin** and PSNE
- ► Mixed Strategies
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MSNE Characterization Theorem



Theorem

A mixed strategy profile is an MSNE iff $\forall i \in N$

- \bullet $u_i(s_i, \sigma_{-i}^*)$ is identical $\forall s_i \in \delta(\sigma_i^*)$,
- $u_i(s_i, \sigma_{-i}^*) \geqslant u_i(s_i', \sigma_{-i}^*), \forall s_i \subseteq \delta(\sigma_i^*), s_i' \notin \delta(\sigma_i^*).$

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Observations:

• $\max_{\sigma_i \in \Delta S_i} u_i(\sigma_i, \sigma_{-i}) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i})$ maximizing w.r.t. a distribution \Leftrightarrow whole probability mass at max

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- If (σ_i^*, σ_i^*) is an MSNE, then

$$\max_{\sigma_i \in \Delta S_i} u_i(\sigma_i, \sigma_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*) = \max_{s_i \in \delta(\sigma_i^*)} u_i(s_i, \sigma_{-i}^*)$$

the maximizer must lie in $\delta(\sigma_i^*)$ – if not, then put all probability mass on that $s_i' \notin \delta(\sigma_i^*)$ that has the maximum value of the utility – $(\sigma_i^*, \sigma_{-i}^*)$ is not a MSNE

Proof of MSNE Characterization Theorem



 (\Rightarrow) Given $(\sigma_i^*, \sigma_{-i}^*)$ is an MSNE

$$u_i(\sigma_i^*, \sigma_{-i}^*) = \max_{\sigma_i \in \Delta S_i} u_i(\sigma_i, \sigma_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*) = \max_{s_i \in \delta(\sigma_i^*)} u_i(s_i, \sigma_{-i}^*)$$
(1)

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By definition of expected utility

$$u_i(\sigma_i^*, \sigma_{-i}^*) = \sum_{s_i \in S_i} \sigma_i^*(s_i) u_i(s_i, \sigma_{-i}^*) = \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i) u_i(s_i, \sigma_{-i}^*)$$
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(2)

Equations (1) and (2) are equal, i.e., max is equal to positive weighted average – can happen only when all values are same: proves condition 1



For **condition 2**: Suppose for contradiction, there exists $s_i \in \delta(\sigma_i^*)$ and $s_i' \notin \delta(\sigma_i^*)$ s.t. $u_i(s_i, \sigma_{-i}^*) < u_i(s_i', \sigma_{-i}^*)$



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We can shift the probability mass $\sigma^*(s_i)$ to s_i' , this new mixed strategy gives a strict higher utility to player i: contradicts MSNE



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This completes the proof of the necessary direction.



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 (\Leftarrow) Given the 2 conditions of the theorem, need to show that $(\sigma_i^*, \sigma_{-i}^*)$ is an MSNE



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 (\Leftarrow) Given the 2 conditions of the theorem, need to show that $(\sigma_i^*, \sigma_{-i}^*)$ is an MSNE

Let
$$u_i(s_i, \sigma_{-i}^*) = m_i(\sigma_{-i}^*), \forall s_i \in \delta(\sigma_i^*)$$
 condition 1
Note $m_i(\sigma_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*)$ condition 2



$$u_i(\sigma_i^*, \sigma_{-i}^*) = \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i) u_i(s_i, \sigma_{-i}^*),$$

by definition of $\delta(\sigma_i^*)$



$$u_{i}(\sigma_{i}^{*}, \sigma_{-i}^{*}) = \sum_{s_{i} \in \delta(\sigma_{i}^{*})} \sigma_{i}^{*}(s_{i}) u_{i}(s_{i}, \sigma_{-i}^{*}),$$

= $m_{i}(\sigma_{-i}^{*})$

by definition of $\delta(\sigma_i^*)$ previous conclusion



$$u_{i}(\sigma_{i}^{*}, \sigma_{-i}^{*}) = \sum_{s_{i} \in \delta(\sigma_{i}^{*})} \sigma_{i}^{*}(s_{i}) u_{i}(s_{i}, \sigma_{-i}^{*}),$$

$$= m_{i}(\sigma_{-i}^{*})$$

$$= \max_{s_{i} \in S_{i}} u_{i}(s_{i}, \sigma_{-i}^{*})$$

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$$\begin{split} u_i(\sigma_i^*,\sigma_{-i}^*) &= \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i) u_i(s_i,\sigma_{-i}^*), \\ &= m_i(\sigma_{-i}^*) \\ &= \max_{s_i \in S_i} u_i(s_i,\sigma_{-i}^*) \\ &= \max_{\sigma_i \in \Delta S_i} u_i(\sigma_i,\sigma_{-i}^*) \end{split}$$

by definition of $\delta(\sigma_i^*)$

previous conclusion previous conclusion

from the observation



$$\begin{split} u_{i}(\sigma_{i}^{*},\sigma_{-i}^{*}) &= \sum_{s_{i} \in \delta(\sigma_{i}^{*})} \sigma_{i}^{*}(s_{i}) u_{i}(s_{i},\sigma_{-i}^{*}), \\ &= m_{i}(\sigma_{-i}^{*}) \\ &= \max_{s_{i} \in S_{i}} u_{i}(s_{i},\sigma_{-i}^{*}) \\ &= \max_{\sigma_{i} \in \Delta S_{i}} u_{i}(\sigma_{i},\sigma_{-i}^{*}) \\ &\geqslant u_{i}(\sigma_{i},\sigma_{-i}^{*}), \ \forall \sigma_{i} \in \Delta S_{i} \end{split}$$

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$$u_{i}(\sigma_{i}^{*}, \sigma_{-i}^{*}) = \sum_{s_{i} \in \delta(\sigma_{i}^{*})} \sigma_{i}^{*}(s_{i}) u_{i}(s_{i}, \sigma_{-i}^{*}),$$

$$= m_{i}(\sigma_{-i}^{*})$$

$$= \max_{s_{i} \in S_{i}} u_{i}(s_{i}, \sigma_{-i}^{*})$$

$$= \max_{\sigma_{i} \in \Delta S_{i}} u_{i}(\sigma_{i}, \sigma_{-i}^{*})$$

$$\geq u_{i}(\sigma_{i}, \sigma_{-i}^{*}), \forall \sigma_{i} \in \Delta S_{i}$$

by definition of $\delta(\sigma_i^*)$

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from the observation

This proves the sufficient direction. The result yields an algorithmic way to find MSNE

Contents



- ► Matrix games
- ▶ Relation between **maxmin** and PSNE
- ► Mixed Strategies
- ► Mixed Strategy Nash Equilibrium
- ► Find MSNE
- ► MSNE Characterization Theorem Proof
- ► Algorithm to find MSNE
- ► Existence of MSNE

MSNE characterization theorem to algorithm



Consider a NFG $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$

MSNE characterization theorem to algorithm



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The total number of supports of
$$S_1 \times S_2 \times S_3 \cdots \times S_n$$
 is $K = (2^{|S_1|} - 1) \times (2^{|S_2|} - 1) \times \cdots \times (2^{|S_n|} - 1)$

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For every support profile $X_1 \times X_2 \times \cdots \times X_n$, where $X_i \subseteq S_i$, solve the following feasibility program

Program

$$\begin{split} w_i &= \sum_{s_{-i} \in S_{-i}} (\prod_{j \neq i} \sigma_j(s_j)) \cdot u_i(s_i, s_{-i}), \forall s_i \in X_i, \forall i \in N \\ w_i &\geqslant \sum_{s_{-i} \in S_{-i}} (\prod_{j \neq i} \sigma_j(s_j)) \cdot u_i(s_i, s_{-i}), \forall s_i \in S_i \setminus X_i, \forall i \in N \\ \sigma_j(s_j) &\geqslant 0, \forall s_j \in S_j, \forall j \in N, \qquad \sum_{s_j \in X_i} \sigma_j(s_j) = 1, \forall j \in N \end{split}$$

Remarks on the algorithm



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• This is not a linear program unless n = 2

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- This is not a linear program unless n = 2
- For general game, there is no poly-time algorithm
- Problem of finding an MSNE is PPAD-complete [Polynomial Parity Argument on Directed graphs] ¹

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MSNE and Dominance



The previous algorithm can be applied to a smaller set of strategies by removing the dominated strategies

Is there a dominated strategy in this game? Domination can be via mixed strategies too

	Player 2	
	L	R
T	4,1	2 , 5
M M	1,3	6,2
В	2,2	3,3

MSNE and Dominance



Theorem

If a pure strategy s_i is strictly dominated by a mixed strategy $\sigma_i \in \Delta S_i$, then in every MSNE of the game, s_i is chosen with probability zero.

So, We can remove such strategies without loss of equilibrium

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Definition (Finite Games)

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Proof requires a few tools and a result from real analysis. Proof is separately given in the course webpage.



Some background for understanding the proof.

• A set $S \subseteq \mathbb{R}^n$ is **convex** if $\forall x, y \in S$ and $\forall \lambda \in [0,1]$, $\lambda x + (1 - \lambda)y \in S$.



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A result from real analysis (proof omitted):

Brouwer's fixed point theorem

If $S \subseteq \mathbb{R}^n$ is **convex** and **compact** and $T: S \to S$, is **continuous** then T has a fixed point, i.e., $\exists \ x^* \in S \text{ s.t. } T(x^*) = x^*.$



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