CS 6001: Game Theory and Algorithmic Mechanism Design

Week 4

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Slide preparation acknowledgments: Onkar Borade and Rounak Dalmia

ज्ञानम् परमम् ध्येयम्
Knowledge is the supreme goal

Contents

Recap

- MSNE → weakest notion of equilibrium so far
- Existence is guaranteed for finite games
- Finding MSNE is computationally expensive

Contents

Correlated Strategy and Equilibrium

Alternative approach - entry of a **mediating** agent/device

Why do we need such an agent?

- Alternative explanation of player rationality
- Utility for all players may get better
- Computational tractability

Correlated Strategy and Equilibrium

		Player 2	
		Wait	Go
Player 1	Wait	0, 0	1, 2
	Go	2, 1	$-10, -10$

Busy cross road game

Nash solutions for the above are

- One waits and the other goes, or
- Large probability of waiting

Correlated Strategy and Equilibrium

- In practice, something else happens
- A traffic light guides the players, and the players agree to this plan (**Why?**)
- The **trusted third party** is called the **mediator**
- **Role**:
  - randomize over the **strategy profiles** (and not individual strategies)
  - and suggest the corresponding strategies to the players
- If the strategies are **enforceable** then it is an equilibrium (**correlated**)

Correlated Strategy and Equilibrium (contd.)

Definition (Correlated Strategy)

A **correlated strategy** is a mapping $\pi : S_1 \times S_2 \times \cdots \times S_n \to [0, 1]$ s.t. $\sum_{s \in S} \pi(s) = 1$.

Example: $\pi(W, W) = 0$, $\pi(W, G) = \pi(G, W) = \frac{1}{2}$, and $\pi(G, G) = 0$

Question

What is a correlated equilibrium?

Answer

A *correlated strategy* is a **correlated equilibrium** when no player *gains* by deviating from the suggested strategy while others follow the suggested strategies

The correlated strategy π *is a common knowledge*

Correlated Strategy and Equilibrium (contd.)

Definition (Correlated Equilibrium)

A **correlated equilibrium** is a correlated strategy π s.t.

$$\sum_{s_{-i} \in S_{-i}} \pi(s_i, s_{-i}) \cdot u_i(s_i, s_{-i}) \geqslant \sum_{s_{-i} \in S_{-i}} \pi(s_i, s_{-i}) \cdot u_i(s_i', s_{-i}), \; \forall s_i, s_i' \in S_i, \forall i \in N.$$

Discussions:

- The mediator suggests the actions after running its randomization device π
- Every agent's best response is to follow it if others are also following it

Some examples (upcoming)

Examples

		Friend 2	
		F	C
Friend 1	F	$2, 1$	$0, 0$
	C	$0, 0$	$1, 2$

Football or Cricket Game

MSNE: $\left(\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right)\right)$

Question

Is $\pi(C, C) = \frac{1}{2} = \pi(F, F)$ a CE?

Yes!

Expected utility: MSNE = $\frac{2}{3}$, CE = $\frac{3}{2}$

		Car 2	
		Wait	Go
Car 1	Wait	$0, 0$	$1, 2$
	Go	$2, 1$	$-10, -10$

Busy Cross road

What are the MSNEs?

Question

$\pi(W, G) = \pi(W, W) = \pi(G, W) = \frac{1}{3}$ a CE?

Question

Are there other CEs of this game?

Contents

Computing Correlated Equilibrium

CE finding is to solve a set of linear equations

Two set of constraints

$$\sum_{s_{-i} \in S_{-i}} \pi(s_i, s_{-i}) \cdot u_i(s_i, s_{-i}) \geqslant \sum_{s_{-i} \in S_{-i}} \pi(s_i, s_{-i}) \cdot u_i(s_i', s_{-i}), \forall s_i, s_i' \in S_i, \forall i \in N$$

Total number of inequalities = $O(n \cdot m^2)$, assuming $|S_i| = m$, $\forall i \in N$

$$\pi(s) \geqslant 0, \forall s \in S, \quad \sum_{s \in S} \pi(s) = 1$$

m^n inequalities

Computing Correlated Equilibrium (contd.)

- The inequalities together represent a **feasibility linear program** that is easier to compute than MSNE
- **MSNE** : total number of support profiles = $O(2^{mn})$
- **CE** : number of inequalities $O(m^n)$: exponentially smaller than the above [1]
- Moreover, this can also be used to optimize some objective function, e.g., maximize the sum of utilities of the players

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[1] take log of both quantities to understand this point

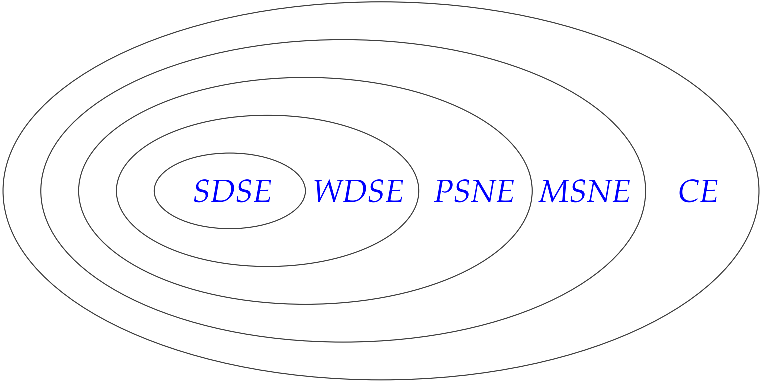
Comparison with the previous equilibrium notions

Theorem

*For every **MSNE** σ^*, there exists a **CE** π^**

Proof Hint: Use $\pi^*(s_i, \ldots, s_n) = \prod_{i=1}^{n} \sigma_i^*(s_i)$ and the MSNE characterization theorem [**Homework**]

Venn diagram of games having equilibrium



Summary so far

- Normal form games
- Rationality, intelligence, common knowledge
- Strategy and action
- Dominance - strict and weak - equilibrium : SDSE, WDSE
- Unilateral deviation - PSNE, generalization : MSNE, existence (Nash)
- Characterization of MSNE - computing, hardness
- Trusted mediator - correlated strategies - equilibrium

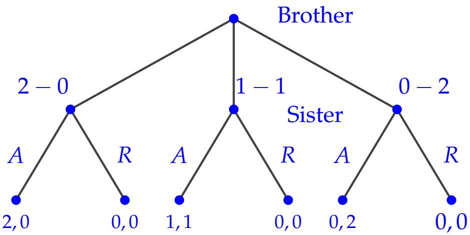
Richer representation of games

- More appropriate for multi-stage games, e.g. **chess**
- Players interact in a sequence - the sequence of actions is the history of the game

Contents

Perfect Information Extensive Games (PIEFG)

- Brother-sister Chocolate Division
- **Disagreement** → both chocolates taken away

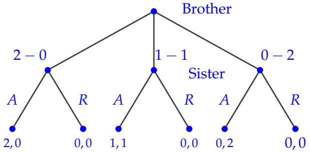


Perfect Information Extensive Form Games (PIEFG)

Formal capture

$$\text{PIEFG } \langle N, A, H, X, P, (u_i)_{i \in N} \rangle$$

- N: a set of players
- A: a set of all possible actions (of all players)
- H: a set of all **sequences of actions** satisfying
  - — empty history $\emptyset \in H$
  - — if $h \in H$, any sub-sequence h' of h starting at the root must be in H
  - — a history $h = (a^{(0)}, a^{(1)}, \ldots, a^{(T-1)})$ is **terminal** if $\nexists\, a^{(T)} \in A$ s.t. $(a^{(0)}, a^{(1)}, \ldots, a^{(T)}) \in H$
  - — $Z \subseteq H$: set of all terminals histories
- $X : H \setminus Z \to 2^A$: action set selection function
- $P : H \setminus Z \to N$: player function
- $u_i : Z \to \mathbb{R}$: utility of i



Perfect Information Extensive Form Games (PIEFG)

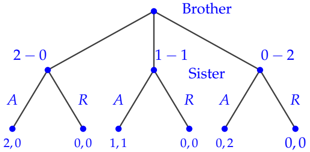
The **strategy** of a player in an EFG is a tuple of actions at every history where the player plays, i.e.,

$$S_i = \times_{\{h \in H : P(h) = i\}} X(h)$$

Remember:

- Strategy is a **complete contingency plan** of the player
- It enumerates potential actions a player can take at every node where she can play, even though some combination of actions may never be executed together

Perfect Information Extensive Form Games (PIEFG)



- $N = \{1, 2\}$ – Brother and Sister respectively
- $A = \{2-0, 1-1, 0-2, A, R\}$
- $H = \{\emptyset, (2-0), (1-1), (0-2), (2-0, A), (2-0, R), (1-1, A), (1-1, R), (0-2, A), (0-2, R)\}$
- $Z = \{(2-0, A), (2-0, R), (1-1, A), (1-1, R), (0-2, A), (0-2, R)\}$
- $X(\emptyset) = \{(2-0), (1-1), (0-2)\}$
- $X(2-0) = X(1-1) = X(0-2) = \{A, R\}$
- $P(\emptyset) = 1, P(2-0) = P(1-1) = P(0-2) = 2$

- $u_1(2-0, A) = 2, u_1(1-1, A) = 1, u_2(1-1, A) = 1, u_2(0-2, A) = 2$ [utilities are zero at the other terminal histories]
- $S_1 = \{2-0, 1-1, 0-2\}$
- $S_2 = \{A, R\} \times \{A, R\} \times \{A, R\} = \{AAA, AAR, ARA, ARR, RAA, RAR, RRA, RRR\}$

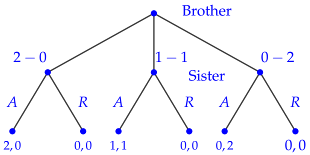
Transforming PIEFG into NFG

Once we have the S_1 and S_2, the game can be represented as an NFG

		Sister							
		AAA	AAR	ARA	ARR	RAA	RAR	RRA	RRR
Brother	2-0	2, 0	2, 0	2, 0	2, 0	0, 0	0, 0	0, 0	0, 0
	1-1	1, 1	1, 1	0, 0	0, 0	1, 1	1, 1	0, 0	0, 0
	0-2	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0

Transforming PIEFG into NFG

		Sister							
		AAA	AAR	ARA	ARR	RAA	RAR	RRA	RRR
Brother	2-0	$2, 0$	$2, 0$	$2, 0$	$2, 0$	$0, 0$	$0, 0$	$0, 0$	$0, 0$
	1-1	$1, 1$	$1, 1$	$0, 0$	$0, 0$	$1, 1$	$1, 1$	$0, 0$	$0, 0$
	0-2	$0, 2$	$0, 0$	$0, 2$	$0, 0$	$0, 2$	$0, 0$	$0, 2$	$0, 0$

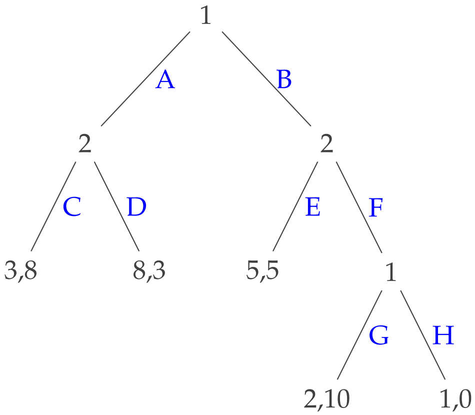


- Nash equilibrium like $(2-0, RRA)$ not quite reasonable, e.g., why R at $1-1$?
- Similarly, $(2-0, RRR)$ is not a **credible threat**, i.e., if the game ever reaches the history $1-1$, Player 2's rational choice is not R
- Hence this equilibrium concept (PSNE) is not good enough for predicting outcomes in PIEFGs
- Also the representation of a sequential game as NFG has huge redundancy – EFG is succinct

Contents

PIEFG to NFG

Equilibrium guarantees are weak for PIEFG in an NFG representation



- Strategies of Player 1 : AG, AH, BG, BH
- Strategies of Player 2 : CE, CF, DE, DF
- PSNEs?
- (AG, CF), (AH, CF), (BH, CE) – is there any non-credible threat
- Better notion of rational outcome will be that which considers a history and ensures utility maximization

Subgame and subgame perfection

Subgame: Game rooted at an intermediate vertex

Definition (Subgame)

The subgame of a PIEFG G rooted at a history h is the *restriction* of G to the descendants of h.

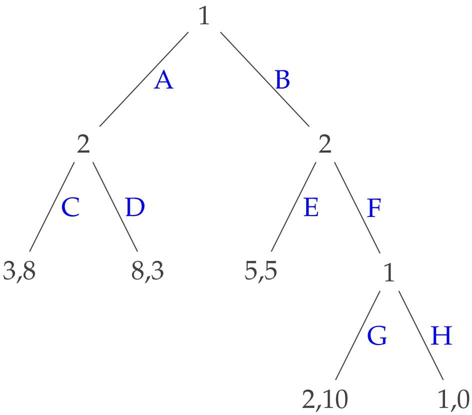
The set of subgames of G is the collection of all subgames at some history of G

Subgame perfection: Best response at every subgame

Definition (Subgame Perfect Nash Equilibrium (SPNE))

A subgame perfect Nash Equilibrium (SPNE) of a PIEFG G is a strategy profile $s \in S$ s.t. for every subgame G' of G, the restriction of s to G' is a PSNE of G'

Example



- PSNEs : (AH, CF), (BH, CE), (AG, CF)
- Are they all SPNEs?
- How to compute them?

Subgame Perfection

Algorithm 1: Backward Induction

```
Function $BACK\_IND(history\ h)$:
    if $h \in Z$ then
        return $u(h), \emptyset$
    $best\_util_{P(h)} \longleftarrow -\infty$
    foreach $a \in X(h)$ do
        $util\_at\_child_{P(h)} \longleftarrow BACK\_IND((h, a))$
        if $util\_at\_child_{P(h)} > best\_util_{P(h)}$ then
            $best\_util_{P(h)} \longleftarrow util\_at\_child_{P(h)}, best\_action_{P(h)} \longleftarrow a$
    return $best\_util_{P(h)}, best\_action_{P(h)}$
```

Contents

Limitations of SPNE

The idea of subgame perfection inherently is based on backward induction

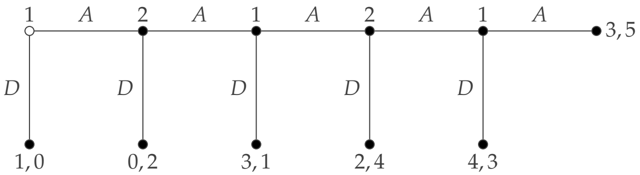
Advantages:

- SPNE is guaranteed to exist in finite PIEFGs (requires proof)
- An SPNE is a PSNE: found a class of games where PSNE is guaranteed to exist
- The algorithm to find SPNE is quite simple

Disdvantages and criticisms:

- The whole tree has to be parsed to find the SPNE: which can be computationally expensive (or maybe impossible), e.g., chess has $\sim 10^{150}$ vertices
- Cognitive limit of real players may prohibit playing an SPNE

Centipede game



Question

What is/are the SPNE(s) of this game?

Question

What is the problem with that prediction ?

Arguments

- This game has been experimented with various populations
- Random participants, university students, grandmasters, etc.
- Most of the subjects (except grandmasters) continue till a few rounds (and not quit at the first round)
- **Reasons claimed:** altruism, limited computational capacity of individuals, incentive difference
- **Criticism of the defining principle of SPNE:** It talks about “what action if the game reached this history” but the equilibrium in some stage above can show that it “cannot reach that history”
- Works in explaining outcomes in certain games, but there is another way to extend this idea
- Using the idea of **belief** of the players

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