



भारतीय प्रौद्योगिकी संस्थान मुंबई
Indian Institute of Technology Bombay

CS 6001: Game Theory and Algorithmic Mechanism Design

Week 5

Swaprava Nath

Slide preparation acknowledgments: Ramsundar Anandanarayanan and Harshvardhan Agarwal

ज्ञानम् परमम् ध्येयम्

Knowledge is the supreme goal



- ▶ Imperfect Information Extensive Form Games

- ▶ Strategies in IIEFGs

- ▶ Equivalence of strategies in IIEFGs

- ▶ Perfect Recall



The story so far

- Games discussed so far (EFGs) are of perfect information

^a<https://rbc.jhuapl.edu/>



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 - several games have states that are unknown to certain agents, e.g., card games like poker, reconnaissance blind chess^a

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- Games discussed so far (EFGs) are of perfect information
- Every player has perfect knowledge about all the developments in the game until that round
- Limited use in certain setups:
 - several games have states that are unknown to certain agents, e.g., card games like poker, reconnaissance blind chess^a
 - not possible to represent simultaneous move games using EFGs

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Games with Imperfect Information



		Kingdom 2	
		Agri	War
Kingdom 1	Agri	5,5	0,6
	War	6,0	1,1

Neighboring Kingdom's Dilemma

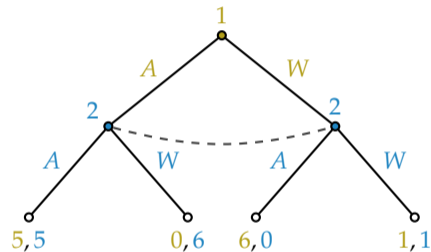
Games with Imperfect Information



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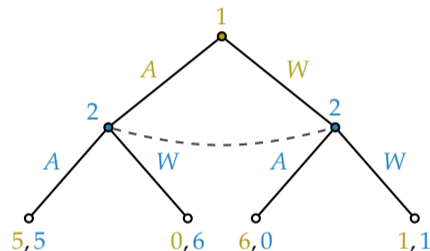
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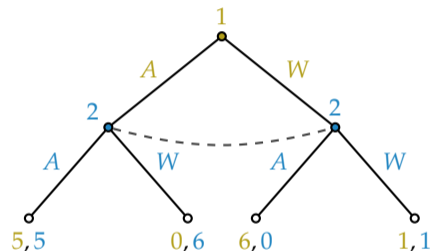
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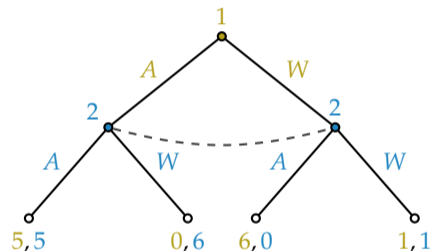


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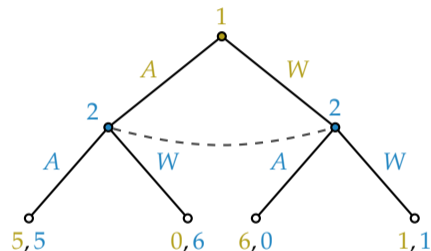
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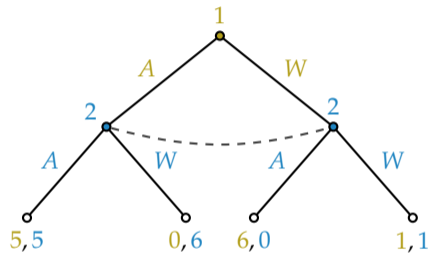
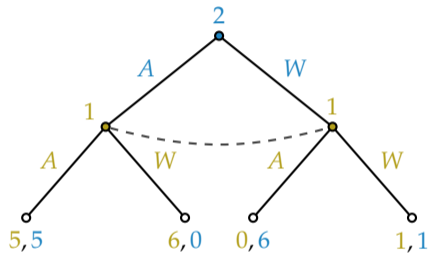
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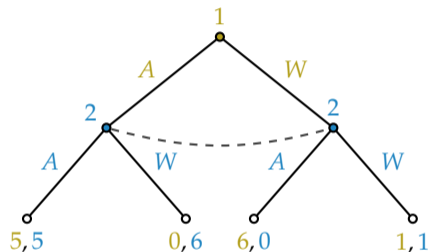
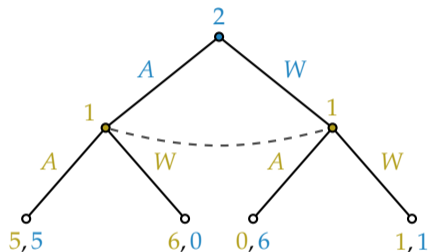


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- Kingdom 2 does not know which node/history the game is in
- These indistinguishable histories form an **information set** for player 2.
- More general representation than PIEFG since information sets can be singleton

Games with Imperfect Information

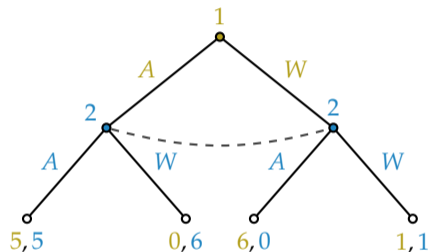
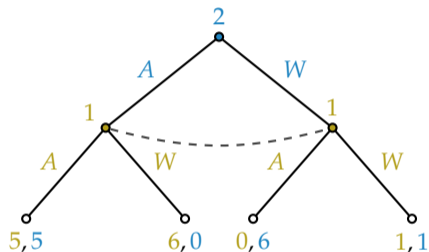


Games with Imperfect Information



- The Neighboring Kingdom's dilemma can also be represented with the information set of player 1 being non-singleton.

Games with Imperfect Information



- The Neighboring Kingdom's dilemma can also be represented with the information set of player 1 being non-singleton.
- IIEFGs are not unique for a given simultaneous move game

Games with Imperfect Information



Definition (IIEFG)

An IIEFG is tuple $\langle N, A, H, X, P, (u_i)_{i \in N}, (I_i)_{i \in N} \rangle$

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Games with Imperfect Information



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- I_i^j s are called an **information set** of player i and I_i is the collection of information sets of i .
- At an information set, the player and her available actions are the same.
- The player is uncertain about which history in the information set is reached.

Games with Imperfect Information (contd.)



Definition (IIIEFG)

An IIIEFG is tuple $\langle N, A, H, X, P, (u_i)_{i \in N}, (I_i)_{i \in N} \rangle$ where $\langle N, A, H, X, P, (u_i)_{i \in N} \rangle$ is a PIEFG and for every $i \in N$, $I_i = (I_i^1, I_i^2, I_i^3, \dots, I_i^{k(i)})$ is a partition of $\{h \in H \setminus Z : P(h) = i\}$ with the property that $X(h) = X(h')$ and $P(h) = P(h') = i$, whenever $\exists j$ s.t. $h, h' \in I_i^j$.

- Some differences with PIEFG

Games with Imperfect Information (contd.)



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- Some differences with PIEFG

- Since actions at an information set are identical, X (action set function) can be defined over I_i^j s i.e.,
$$X(h) = X(h') = X(I_i^j), \forall h, h' \in I_i^j$$

Games with Imperfect Information (contd.)



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- Strategies can also be defined over information sets, i.e., strategy set of a player $i \in N$ is defined as the Cartesian product of actions available to i at her information sets

$$S_i = \times_{I' \in I_i} X(I') = \times_{j=1}^{j=k(i)} X(I_i^j)$$

Games with Imperfect Information (contd.)



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Games with Imperfect Information (contd.)



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- With IIIEFG, NFGs can be represented using EFGs, although not very succinct.
- IIIEFG is a richer representation than both NFG and PIEFG.

Example of Information Addition



- Consider the two-player zero-sum game comprised of the following two stages

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- Each of the following matrices are chosen w.p. $\frac{1}{2}$, but no player sees the realization of this randomization process

		Player II	
		<i>L</i>	<i>R</i>
Player I	<i>T</i>	0	$\frac{1}{2}$
	<i>B</i>	0	1

Matrix G_1

		Player II	
		<i>L</i>	<i>R</i>
Player I	<i>T</i>	1	0
	<i>B</i>	$\frac{1}{2}$	0

Matrix G_2



Example of Information Addition

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		Player II	
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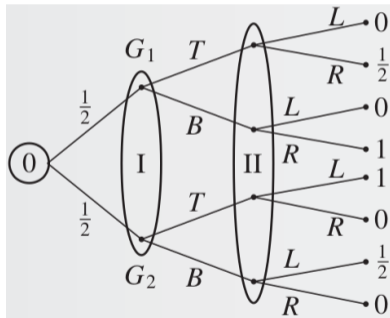
Matrix G_2

- **What is the extensive form representation?**

Example (Contd.)



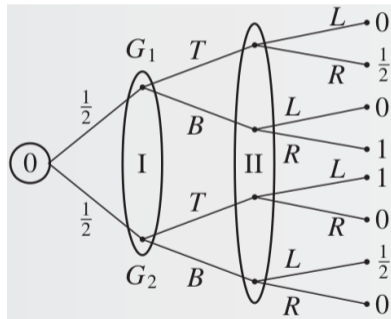
- EFG:





Example (Contd.)

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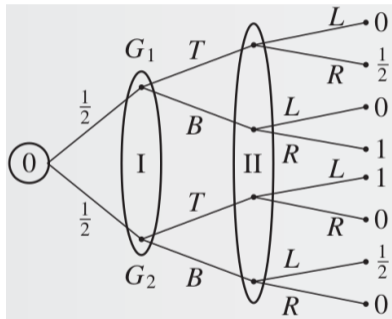


- What is the normal form representation?



Example (Contd.)

- EFG \Rightarrow NFG:

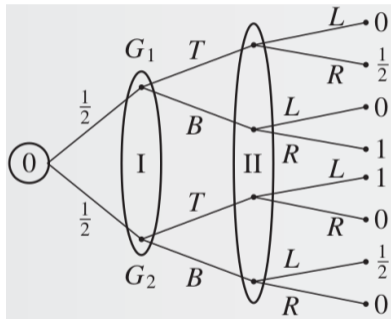


		Player II	
		<i>L</i>	<i>R</i>
Player I	<i>T</i>	$\frac{1}{2}$	$\frac{1}{4}$
	<i>B</i>	$\frac{1}{4}$	$\frac{1}{2}$



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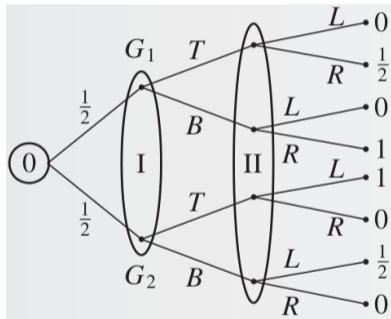
		Player II	
		L	R
Player I	T	$\frac{1}{2}$	$\frac{1}{4}$
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- What is an MSNE of this game?



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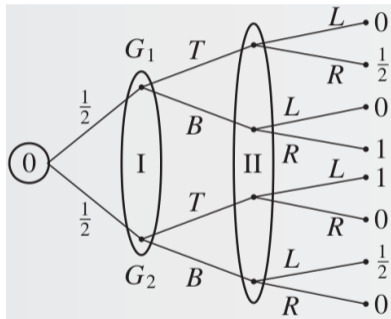
		Player II	
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Player I	T	$\frac{1}{2}$	$\frac{1}{4}$
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- What is an MSNE of this game?
- What is the value of this game?



Example (Contd.)

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Player I	T	$\frac{1}{2}$	$\frac{1}{4}$
	B	$\frac{1}{4}$	$\frac{1}{2}$

- What is an MSNE of this game?
- What is the value of this game?
- MSNE: $\left(\left(\frac{1}{2}, \frac{1}{2} \right), \left(\frac{1}{2}, \frac{1}{2} \right) \right)$, value = $\frac{3}{8}$

Same Example: More Information to Player I

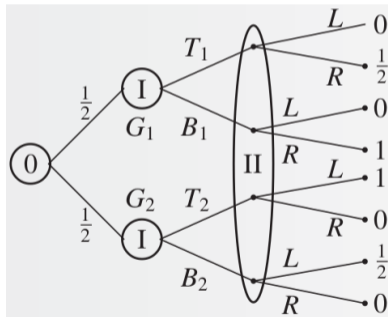


- What happens if Player I is informed (but Player II is not) which matrix was chosen,



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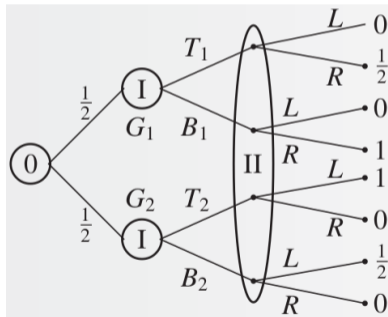
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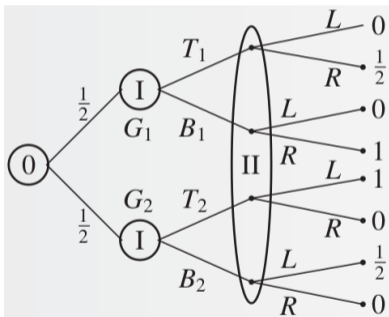


- **What are the strategies now? What is the NFG representation?**

Example (Contd.)



- EFG \Rightarrow NFG:

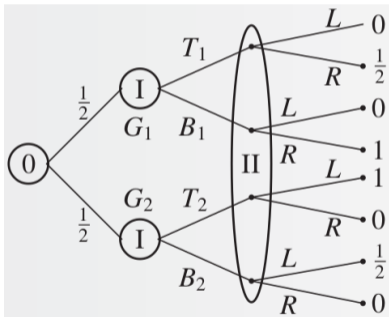


		Player II	
		<i>L</i>	<i>R</i>
Player I	$T_1 T_2$	$\frac{1}{2}$	$\frac{1}{4}$
	$T_1 B_2$	$\frac{1}{4}$	$\frac{1}{4}$
	$B_1 T_2$	$\frac{1}{2}$	$\frac{1}{2}$
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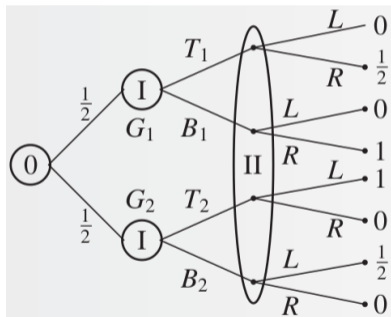
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- What is an MSNE and value of this game?



Example (Contd.)

- EFG \Rightarrow NFG:



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- What is an MSNE and value of this game?
- MSNE: $((1(B_1 T_2)), (p, 1 - p)), p \in [0, 1], \text{value} = \frac{1}{2}$



Theorem

Let Γ be a two-player zero-sum game in extensive form and let Γ' be the game derived from Γ by splitting several information sets of Player I. Then the value of the game Γ' in mixed strategies is greater than or equal to the value of Γ in mixed strategies.

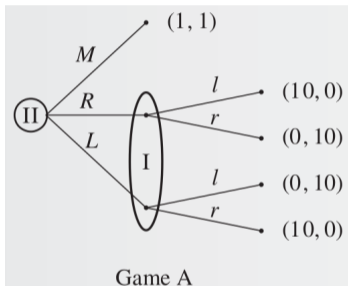


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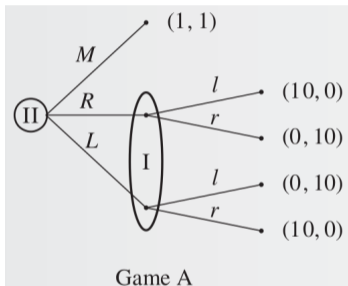
Proof: exercise

How about General-sum Games?



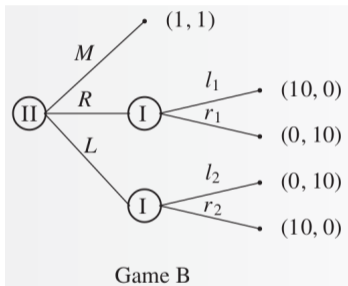
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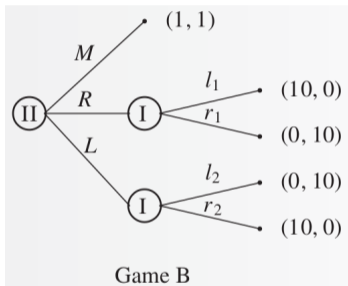
- Find the MSNE of this game!
- $\left(\left(\frac{1}{2}(l), \frac{1}{2}(r) \right), \left(\frac{1}{2}(L), \frac{1}{2}(R), 0(M) \right) \right) \implies \text{expected payoff} = (5, 5)$

Player I gets more information



- Find the MSNE of this game!

Player I gets more information



- Find the MSNE of this game!
- $((1(l_1r_2)), (0(L), 0(R), 1(M))) \implies \text{expected payoff} = (1, 1)$



- ▶ Imperfect Information Extensive Form Games
- ▶ Strategies in IIEFGs
- ▶ Equivalence of strategies in IIEFGs
- ▶ Perfect Recall



- Strategy set of i : $S_i = \times_{j=1}^{j=k(i)} X(I_i^j)$

Randomized Strategies in IIEFGs



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- Strategy set of i : $S_i = \times_{j=1}^{j=k(i)} X(I_i^j)$
- In NFGs, mixed strategies randomize over pure strategies
- In EFGs, randomization can happen in different ways

Randomized Strategies in IIEFGs



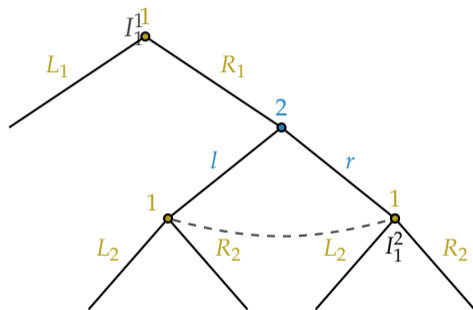
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Randomized Strategies in IIEFGs



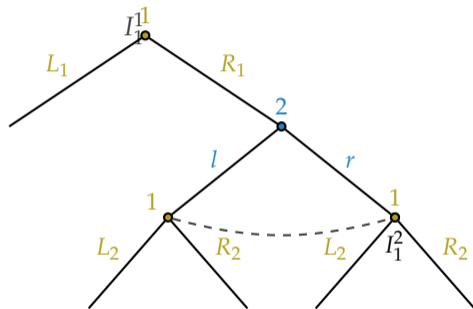
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Randomized Strategies in IIEFGs



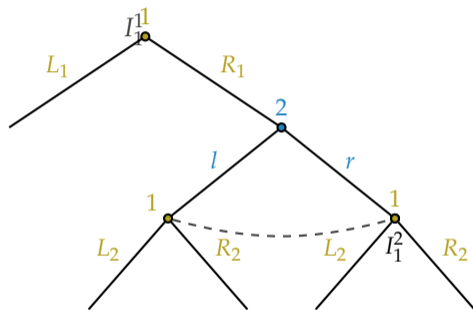
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Randomized Strategies in IIEFGs



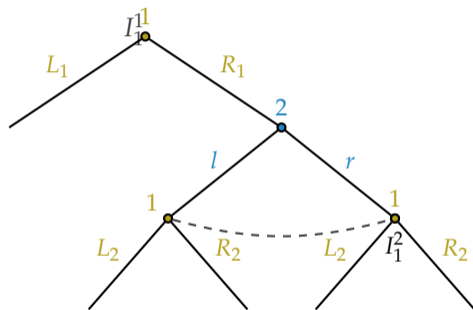
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Randomized Strategies in IIEFGs



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Definition

A **behavioral strategy** of a player in an IIEFG is a function that maps each of her information sets to a probability distribution over the set of possible actions **at that information set**.



Question

What is the relation between mixed and behavioral strategies?



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What is the relation between mixed and behavioral strategies?

- In this example, MSs live in \mathbb{R}^4 , BSs live in two \mathbb{R}^2 spaces
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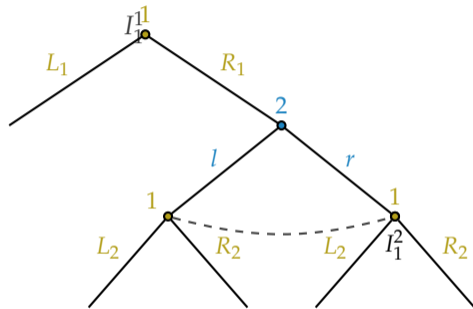
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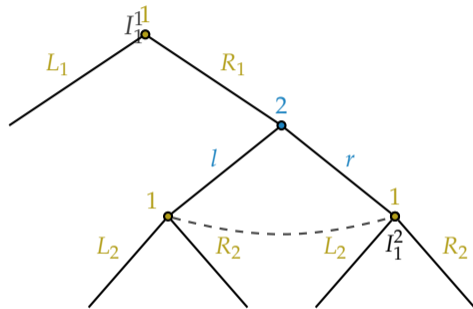
Equivalence in terms of the probability of reaching a vertex/history x

- Say $\rho(x; \sigma)$ is the probability of reaching a node x under mixed strategy profile σ
- Similarly, $\rho(x; b)$ is the same for behavioral strategy profile b

Example

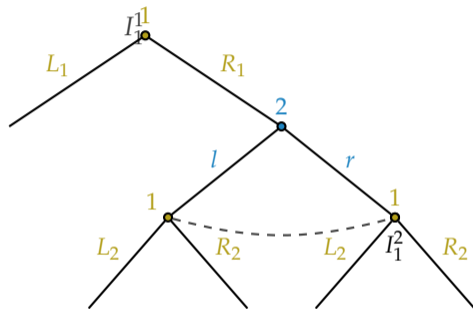


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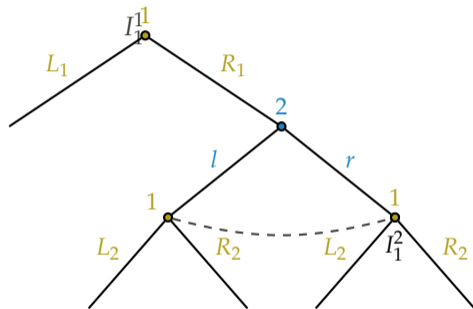
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Example



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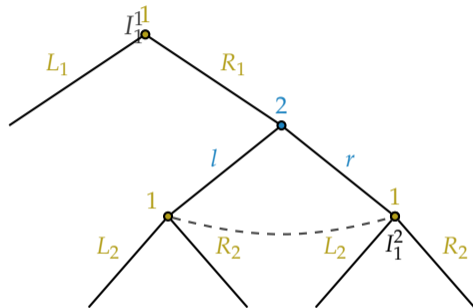
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Players can choose different kind of strategies

$$\rho(x; \sigma_1, b_2) = (\sigma_1(R_1L_2) + \sigma_1(R_1R_2)) \cdot b_2(I_1^2)(r)$$

Equivalence Definition



Definition

A mixed strategy σ_i and a behavioral strategy b_i of a player i in an IIEFG are **equivalent** if for every mixed/behavioral strategy ξ_{-i} of the other players and **every vertex** x in the game tree,

$$\rho(x; \sigma_i, \xi_{-i}) = \rho(x; b_i, \xi_{-i})$$



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Earlier example (right)

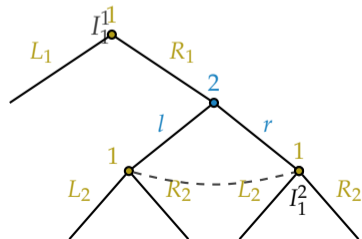
$$b_1(I_1^1)(L_1) = \sigma_1(L_1L_2) + \sigma_1(L_1R_2)$$

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We call b_1 and σ_1 are equivalent.



More on Equivalent Strategies



The equivalence, by definition, holds at the leaf nodes too

More on Equivalent Strategies



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It is enough to check the equivalence only at the leaf nodes.

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Claim

It is enough to check the equivalence only at the leaf nodes.

Reason: Pick an arbitrary non-leaf node, the probability of reaching that node is equal to the sum of the probabilities of reaching the leaf nodes in its subtree.

More on Equivalent Strategies



This argument can be extended further

Theorem (Utility Equivalence)

If σ_i and b_i are equivalent, then for every mixed/behavioral strategy vector of the other players ξ_{-i} , the following holds,

$$u_j(\sigma_i, \xi_{-i}) = u_j(b_i, \xi_{-i}), \quad \forall j \in N.$$

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Repeat the argument for any equivalent mixed and behavioral strategy profiles.

Corollary

Let σ and b be equivalent, i.e., σ_i and b_i are equivalent $\forall i \in N$, then $u_i(\sigma) = u_i(b)$, $\forall i \in N$.



- ▶ Imperfect Information Extensive Form Games
- ▶ Strategies in IIEFGs
- ▶ Equivalence of strategies in IIEFGs
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Equivalence of strategies in IIEFGs



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Why behavioral strategies are desirable?

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Equivalence of strategies in IIEFGs



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Can we construct one from another?

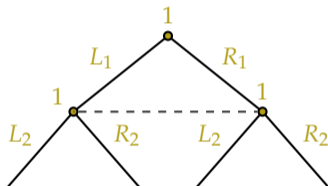
OR

Does equivalence always hold?

Equivalence of strategies in IIEFGs (Example 1)



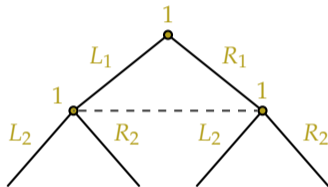
Player remembers that it made a move but forgets which move



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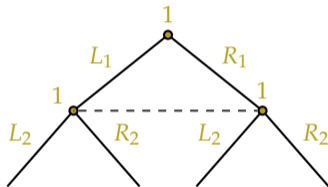


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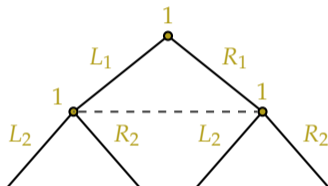


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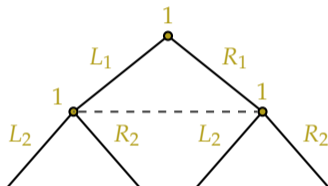


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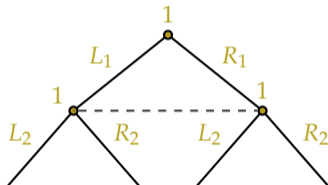


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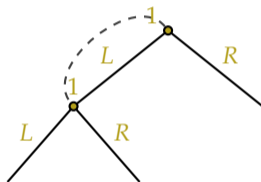


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- **Mixed strategy with no equivalent behavioral strategies**

Equivalence of strategies in IIEFGs (Example 2)



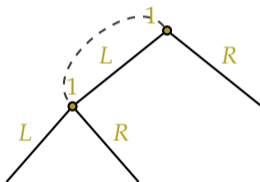
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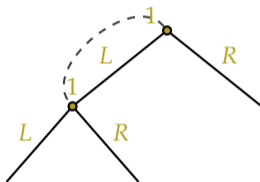


Behavioral strategy: positive mass on the second leaf node possible, not for mixed strategy

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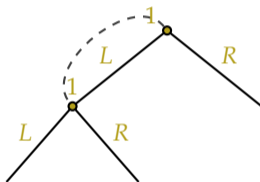
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Answer

The equivalence does not hold if the players are forgetful

Equivalence of strategies in IIEFGs



Question

When does behavioral strategy have no equivalent mixed strategy?

Equivalence of strategies in IIEFGs



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Observation from a graph viewpoint

- 1 Let x be a non-root node



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- 4 If the path from the root to x passes through vertices x_1 and x'_1 that are in the same information set of player i , and the action leading to x at x_1 and x'_1 is different, then no **pure strategy** can ever lead to x

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- 4 If the path from the root to x passes through vertices x_1 and x'_1 that are in the same information set of player i , and the action leading to x at x_1 and x'_1 is different, then no **pure strategy** can ever lead to x
- 5 Since mixed strategy is a randomization over pure strategies, every mixed strategy will put zero probability mass on x but behavioral strategy randomizes on every vertex **independently**, hence x may be reached in behavioral strategies with a positive probability

Equivalence of strategies in IIEFGs



The last observation can be stated as a lemma

Equivalence of strategies in IIEFGs



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Lemma

*If there exists a path from the root to some vertex x that passes through the same information set at least twice, and if the action leading to x is **not** the same at each of those vertices, then the player at the information set has a behavioral strategy that has no equivalent mixed strategy.*

Equivalence of strategies in IIEFGs



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This lemma helps in proving the following characterization result of equivalence.

Equivalence of strategies in IIEFGs



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Theorem (6.11 of MSZ)

Consider an IIEFG such that every vertex has at least two actions. Every behavioral strategy has an equivalent mixed strategy for a player iff each information set of that player intersects every path emanating from the root at most once.

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Proof.

Homework. Reading exercise from MSZ.





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Behavioral Strategy equivalent to Mixed Strategy



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To formalize (i.e., to set the conditions when the equivalence holds), we need to formalize the **forgetfulness** of a player.

- saw few examples of players' forgetfulness.
- our conditions need to ensure that none of the previous types of forgetfulness happens.

Behavioral Strategy equivalent to Mixed Strategy



Definition (Choice of **same action at an information set**)

Let $X = (x^0, x^1, \dots, x^K)$ and $\hat{X} = (x^0, \hat{x}^1, \dots, \hat{x}^L)$ be two paths in the game tree.

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We may use it for a path leading from a node x to some node y which is not an immediate child of x .



Definition (Perfect Recall)

Player i has **perfect recall** if the following conditions are satisfied



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Rephrasing

For every I_i^j of player i and every pair of vertices $x, y \in I_i^j$, if the decision vertices of i are $x_i^1, x_i^2, \dots, x_i^L = x$ and $y_i^1, y_i^2, \dots, y_i^{L'} = y$ respectively for the two paths from the root to x and y , then



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Games with Perfect Recall

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- 1 $L = L'$,
- 2 $x_i^l, y_i^l \in I_i^k$ for some k ,
- 3 $a_i(x_i^l \rightarrow x_i^{l+1}) = a_i(y_i^l \rightarrow y_i^{l+1}), \forall l = 1, 2, \dots, L - 1$.



Definition

A game has **perfect recall** if every player has a perfect recall.

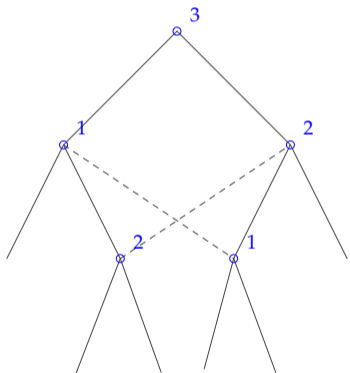


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Note: Definition of perfect recall subsumes the condition where every behavioral strategy has equivalent mixed strategy

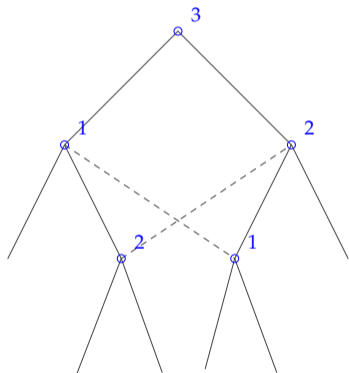
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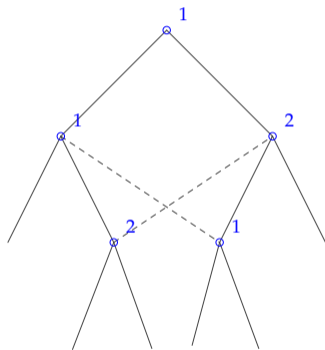


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Game with Perfect Recall: This example satisfies the conditions of the definitions.

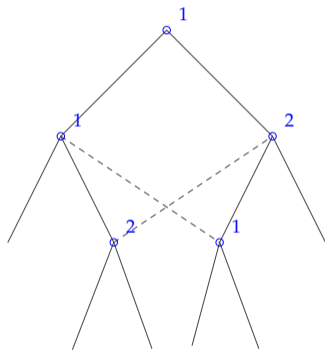
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Game with Imperfect Recall: Player 1 takes two different actions at the first information set to reach two different vertices of the second information set.

Implications of Perfect Recall



Let $S_i^*(x)$ be the set of pure strategies of player i at which he chooses actions leading to x , i.e., intersections of members of S_i with the path from root to x .

Theorem

If i is a player with perfect recall and x and x' are the two vertices in the same information set of i , then $S_i^(x) = S_i^*(x')$.*

The above conclusion comes from the same sequence of information sets and same actions. The next implication of mixed and behavioral strategies.

Implications of Perfect Recall



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In every IIEFG, if i is a player with perfect recall, then for every mixed strategy of i , there exists an equivalent behavioral strategy.



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- The proof is constructive. It starts with the mixed strategy and constructs the behavioral strategies such that the probabilities of reaching a leaf are same. The arguments show that such a construction is always possible because of perfect recall.



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