भारतीय प्रौद्योगिकी संस्थान मुंबई

Indian Institute of Technology Bombay

CS 6001: Game Theory and Algorithmic Mechanism Design

Week 5

Swaprava Nath

Slide preparation acknowledgments: Ramsundar Anandanarayanan and Harshvardhan Agarwal

ज्ञानम् परमम् ध्येयम्

Knowledge is the supreme goal

Contents

Games with Imperfect Information

The story so far

- Games discussed so far (EFGs) are of perfect information

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[a] https://rbc.jhuapl.edu/

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- Games discussed so far (EFGs) are of perfect information
- Every player has perfect knowledge about all the developments in the game until that round
- Limited use in certain setups:
    - several games have states that are unknown to certain agents, e.g., card games like poker, reconnaissance blind chess[a]
    - not possible to represent simultaneous move games using EFGs

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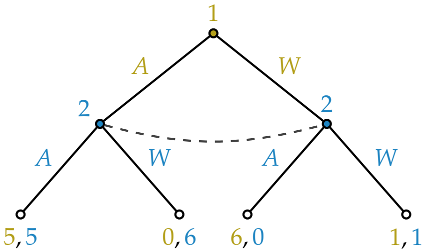
Games with Imperfect Information

		Kingdom 2	
		Agri	War
Kingdom 1	Agri	5, 5	0, 6
	War	6, 0	1, 1

Neighboring Kingdom's Dilemma



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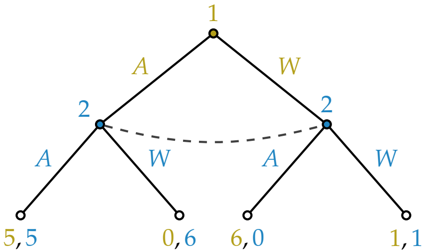

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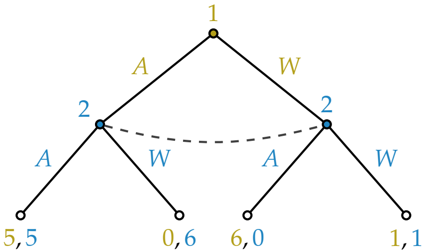

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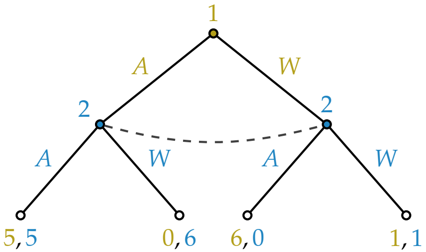

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- These indistinguishable histories form an **information set** for player 2.

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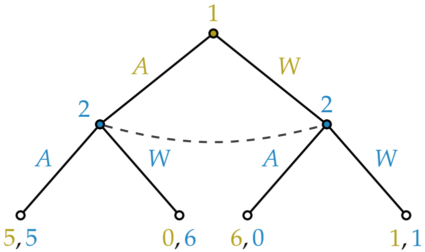
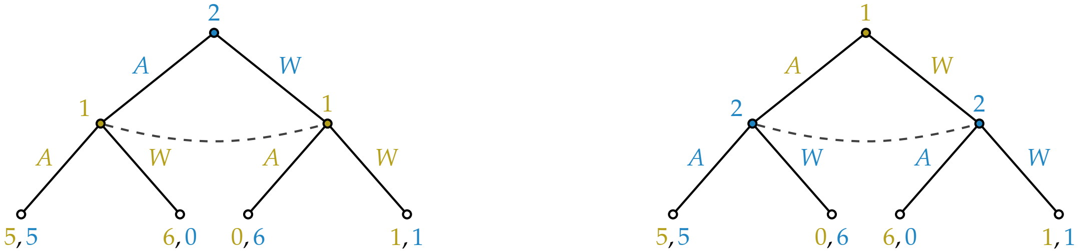


- In IIEFG, indistinguishable nodes are connected via a dotted line.
- Kingdom 2 does not know which node/history the game is in
- These indistinguishable histories form an **information set** for player 2.
- More general representation than PIEFG since information sets can be singleton

Games with Imperfect Information



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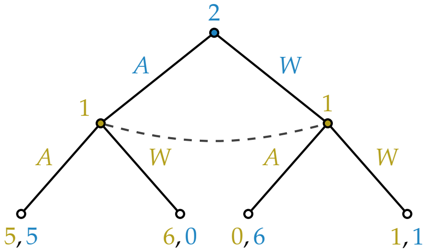
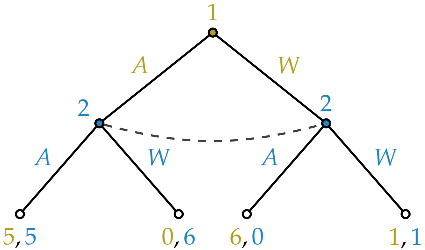
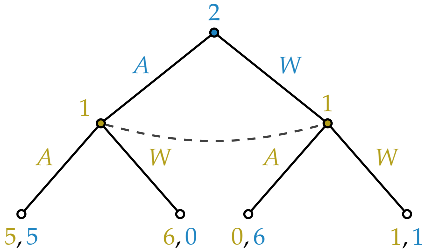
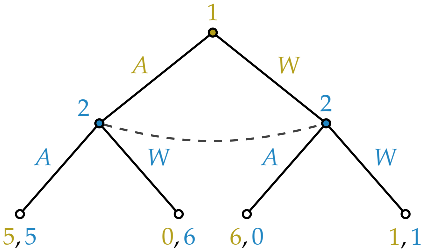




- The Neighboring Kingdom’s dilemma can also be represented with the information set of player 1 being non-singleton.

Games with Imperfect Information





- The Neighboring Kingdom's dilemma can also be represented with the information set of player 1 being non-singleton.
- IIEFGs are not unique for a given simultaneous move game

Games with Imperfect Information

Definition (IIEFG)

An IIEFG is tuple $\langle N, A, H, X, P, (u_i)_{i \in N}, (I_i)_{i \in N} \rangle$

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- I_i^js are called an **information set** of player i and I_i is the collection of information sets of i.
- At an information set, the player and her available actions are the same.
- The player is uncertain about which history in the information set is reached.

Games with Imperfect Information (contd.)

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- Some differences with PIEFG
  - Since actions at an information set are identical, X (action set function) can be defined over $I_i^j s$ i.e., $X(h) = X(h') = X(I_i^j), \forall h, h' \in I_i^j$

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  - Strategies can also be defined over information sets, i.e., strategy set of a player $i \in N$ is defined as the Cartesian product of actions available to i at her information sets $S_i = \times_{I' \in I_i} X(I') = \times_{j=1}^{j=k(i)} X(I_i^j)$

Games with Imperfect Information (contd.)

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- With IIEFG, NFGs can be represented using EFGs, although not very succinct.
- IIEFG is a richer representation than both NFG and PIEFG.

Example of Information Addition

- Consider the two-player zero-sum game comprised of the following two stages

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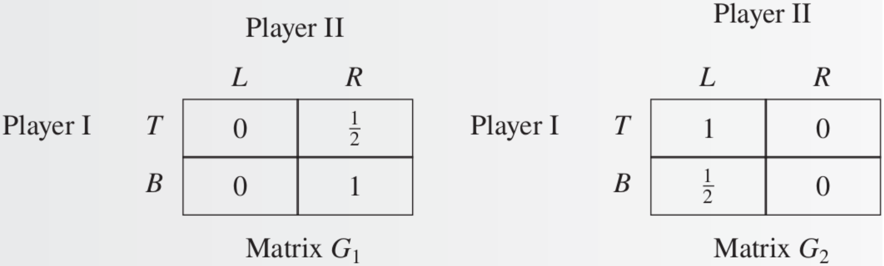
- Each of the following matrices are chosen w.p. $\frac{1}{2}$, but no player sees the realization of this randomization process



$$\begin{array}{c|c|c|c|}
 & & \multicolumn{2}{c}{\text{Player II}} \\
 & & L & R \\
\hline
\text{Player I} & T & 0 & \frac{1}{2} \\
\hline
 & B & 0 & 1 \\
\hline
\end{array}$$

Matrix G_1

$$\begin{array}{c|c|c|c|}
 & & \multicolumn{2}{c}{\text{Player II}} \\
 & & L & R \\
\hline
\text{Player I} & T & 1 & 0 \\
\hline
 & B & \frac{1}{2} & 0 \\
\hline
\end{array}$$

Matrix G_2

Example of Information Addition

- Consider the two-player zero-sum game comprised of the following two stages
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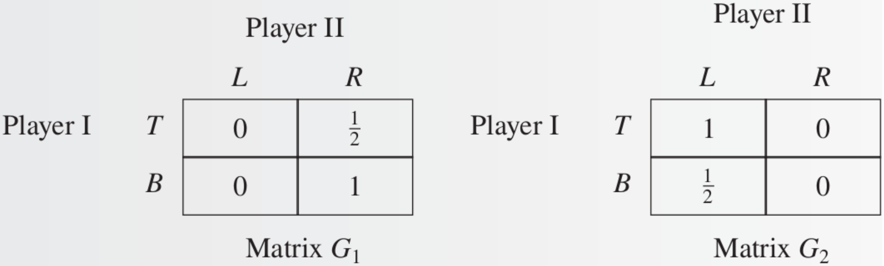


		Player II	
		L	R
Player I	T	0	$\frac{1}{2}$
	B	0	1

Matrix G_1

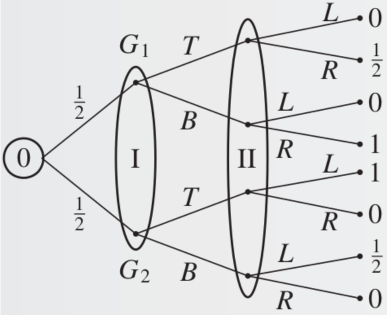
		Player II	
		L	R
Player I	T	1	0
	B	$\frac{1}{2}$	0

Matrix G_2

- **What is the extensive form representation?**

Example (Contd.)

- EFG:



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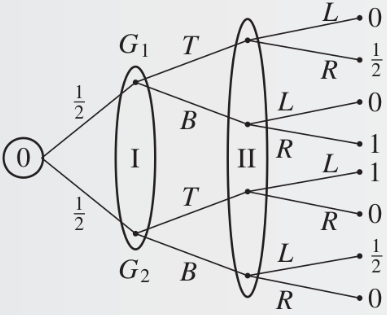
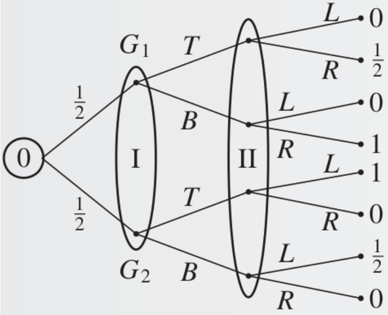
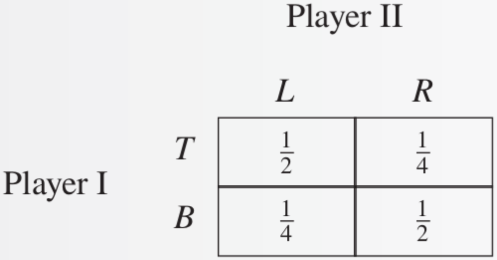


- **What is the normal form representation?**

Example (Contd.)

- EFG ⇒ NFG:

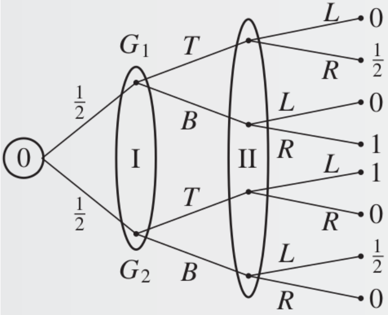


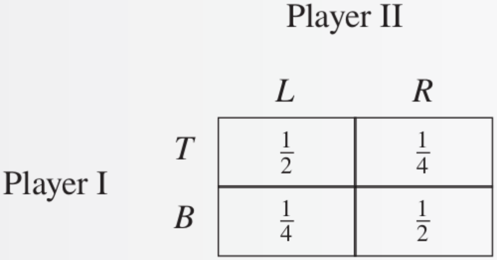


		Player II	
		L	R
Player I	T	$\frac{1}{2}$	$\frac{1}{4}$
	B	$\frac{1}{4}$	$\frac{1}{2}$

Example (Contd.)

- EFG $\Rightarrow$ NFG:





		Player II	
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Player I	T	$\frac{1}{2}$	$\frac{1}{4}$
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- **What is an MSNE of this game?**

Example (Contd.)

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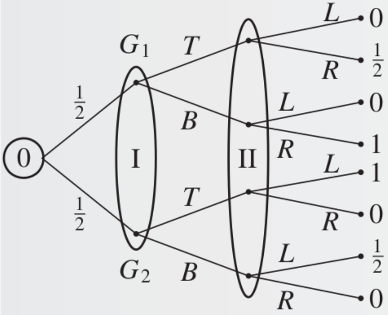

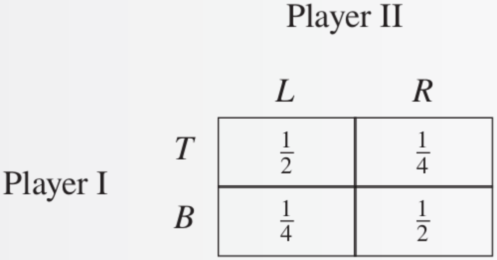

		Player II	
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Player I	T	$\frac{1}{2}$	$\frac{1}{4}$
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- **What is an MSNE of this game?**
- **What is the value of this game?**

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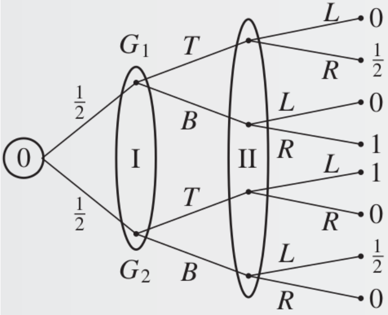

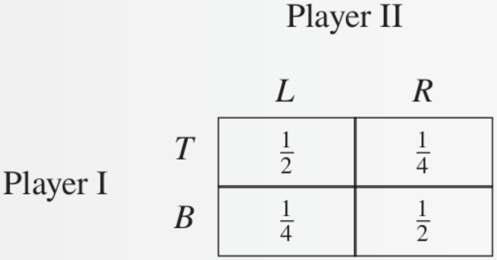

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- **What is an MSNE of this game?**
- **What is the value of this game?**
- **MSNE: $\left(\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)\right)$, value = $\frac{3}{8}$**

Same Example: More Information to Player I

- What happens if Player I is informed (but Player II is not) which matrix was chosen,

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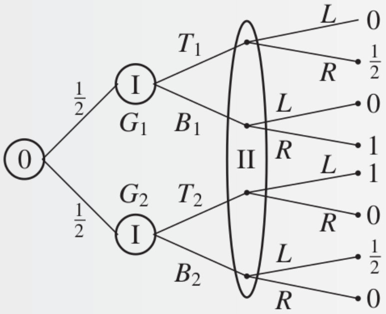

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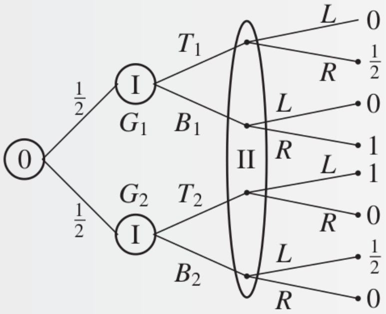
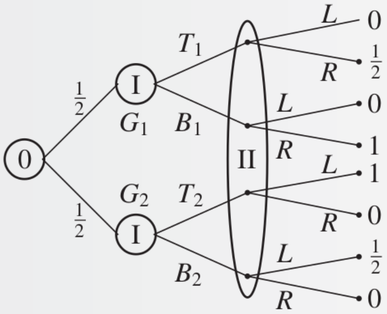
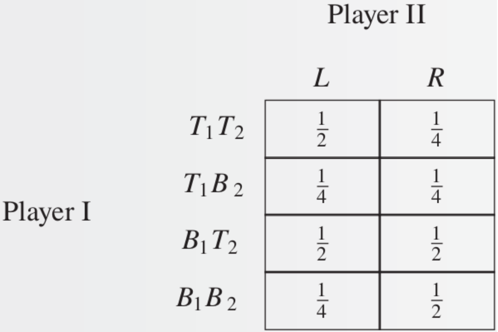


- **What are the strategies now? What is the NFG representation?**

Example (Contd.)

- EFG ⇒ NFG:





		Player II	
		L	R
Player I	$T_1 T_2$	$\frac{1}{2}$	$\frac{1}{4}$
	$T_1 B_2$	$\frac{1}{4}$	$\frac{1}{4}$
	$B_1 T_2$	$\frac{1}{2}$	$\frac{1}{2}$
	$B_1 B_2$	$\frac{1}{4}$	$\frac{1}{2}$

Example (Contd.)

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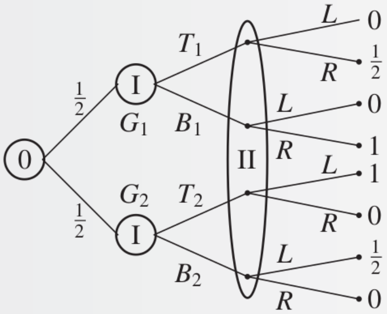
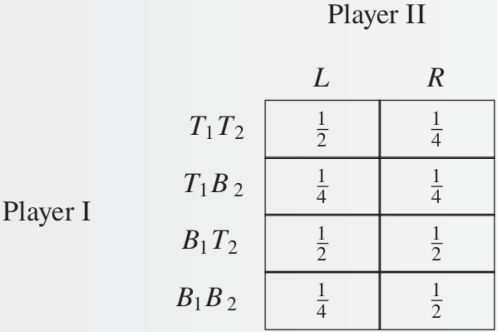


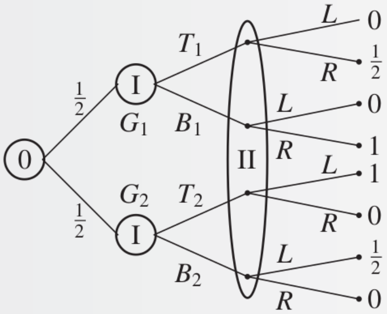


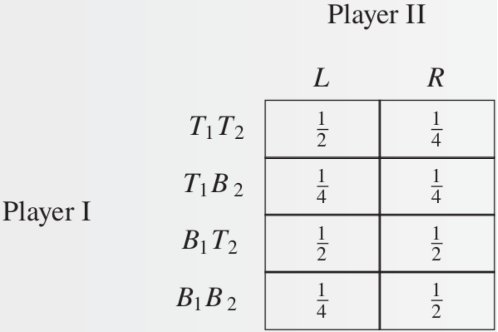
		Player II	
		L	R
Player I	T_1T_2	$\frac{1}{2}$	$\frac{1}{4}$
	T_1B_2	$\frac{1}{4}$	$\frac{1}{4}$
	B_1T_2	$\frac{1}{2}$	$\frac{1}{2}$
	B_1B_2	$\frac{1}{4}$	$\frac{1}{2}$

- **What is an MSNE and value of this game?**

Example (Contd.)

- EFG $\Rightarrow$ NFG:





		Player II	
		L	R
Player I	T_1T_2	$\frac{1}{2}$	$\frac{1}{4}$
	T_1B_2	$\frac{1}{4}$	$\frac{1}{4}$
	B_1T_2	$\frac{1}{2}$	$\frac{1}{2}$
	B_1B_2	$\frac{1}{4}$	$\frac{1}{2}$

- **What is an MSNE and value of this game?**
- **MSNE:** $((1(B_1T_2)),(p,1-p))\,,p\in[0,1]$, **value =** $\frac{1}{2}$

Result on Information Addition in Matrix Games

Theorem

Let Γ be a two-player zero-sum game in extensive form and let Γ' be the game derived from Γ by splitting several information sets of Player I. Then the value of the game Γ' in mixed strategies is greater than or equal to the value of Γ in mixed strategies.

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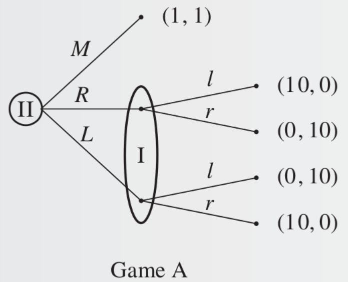
Proof: exercise

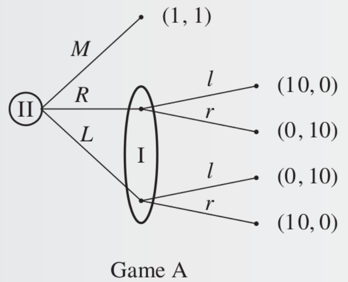
How about General-sum Games?



Game A

- Find the MSNE of this game!

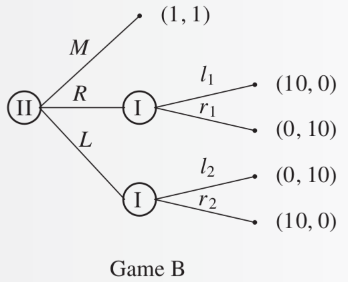
How about General-sum Games?



Game A

- Find the MSNE of this game!
- $\left((\frac{1}{2}(l), \frac{1}{2}(r)), (\frac{1}{2}(L), \frac{1}{2}(R), 0(M)) \right) \implies$ expected payoff = $(5, 5)$

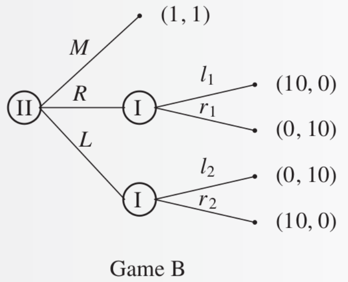
Player I gets more information



Game B

- Find the MSNE of this game!

Player I gets more information



- Find the MSNE of this game!
- $((1(l_1 r_2)), (0(L), 0(R), 1(M))) \implies$ expected payoff = $(1, 1)$

Contents

Randomized Strategies in IIEFGs

- Strategy set of i : $S_i = \times_{j=1}^{j=k(i)} X(I_i^j)$

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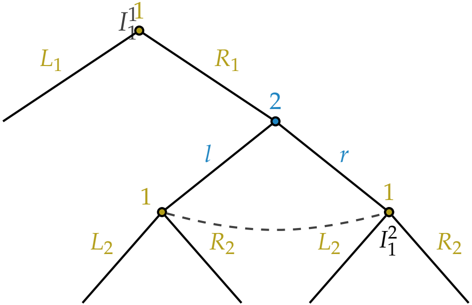
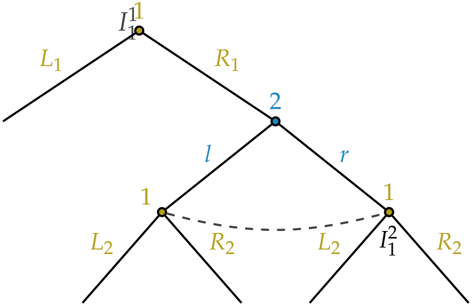
Randomized Strategies in IIEFGs



- Strategies?

Randomized Strategies in IIEFGs



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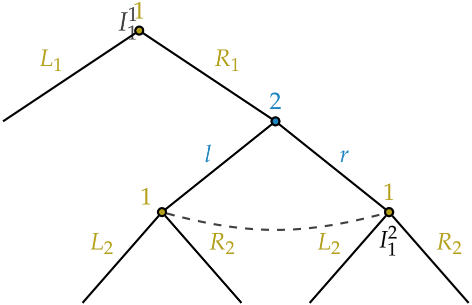

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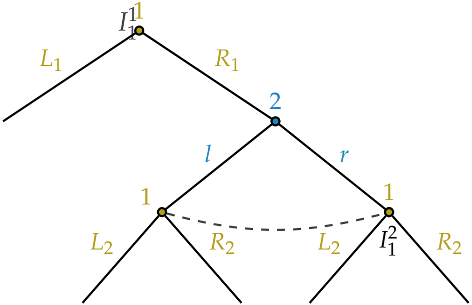

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- Behavioral Strategy b_1, $b_1(I_1^1) \in \Delta(L_1, R_1), b_1(I_1^2) \in \Delta(L_2, R_2), b_2(I_2^1) \in \Delta(l, r)$

Behavioral Strategy

Definition

A **behavioral strategy** of a player in an IIEFG is a function that maps each of her information sets to a probability distribution over the set of possible actions **at that information set**.

Mixed and Behavioral strategy

Question

What is the relation between mixed and behavioral strategies?

Mixed and Behavioral strategy

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What is the relation between mixed and behavioral strategies?

- In this example, MSs live in $\mathbb{R}^4$, BSs live in two $\mathbb{R}^2$ spaces
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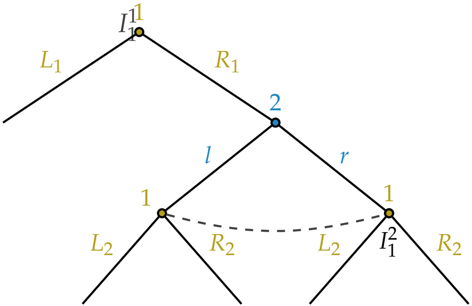
Can we have an equivalence?

Equivalence in terms of the probability of reaching a vertex/history x

- Say $\rho(x;\sigma)$ is the probability of reaching a node x under mixed strategy profile σ
- Similarly, $\rho(x;b)$ is the same for behavioral strategy profile b

Example



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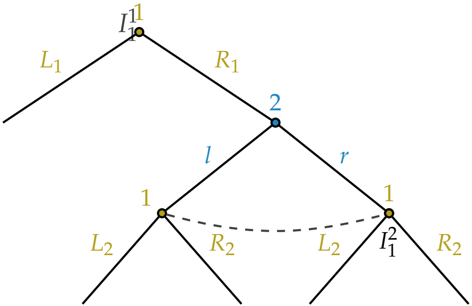

$$\rho(x;\sigma) = \sigma_1(R_1)\sigma_2(r)$$

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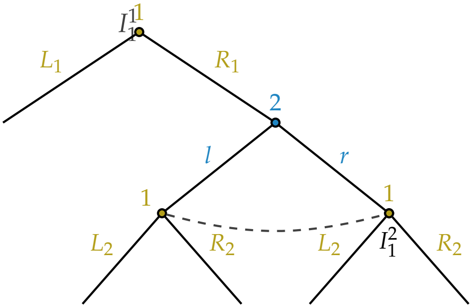

$$
\begin{aligned}
\rho(x;\sigma) &= \sigma_1(R_1)\sigma_2(r) \\
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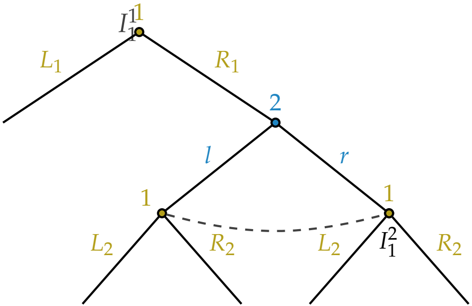

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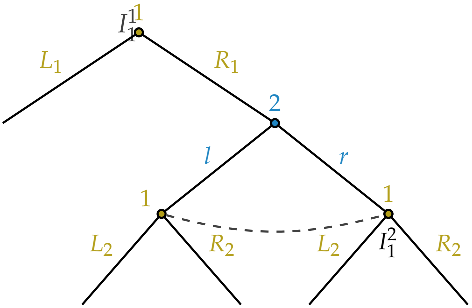

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Players can choose different kind of strategies

$$\rho(x;\sigma_1,b_2) = (\sigma_1(R_1L_2)+\sigma_1(R_1R_2))\cdot b_2(I_2^1)(r)$$

Equivalence Definition

Definition

A mixed strategy σ_i and a behavioral strategy b_i of a player i in an IIEFG are **equivalent** if for every mixed/behavioral strategy ξ_{-i} of the other players and **every vertex** x in the game tree,

$$\rho(x; \sigma_i, \xi_{-i}) = \rho(x; b_i, \xi_{-i})$$

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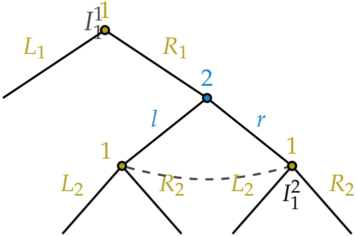
Earlier example (right)

$$b_1(I_1^1)(L_1) = \sigma_1(L_1L_2) + \sigma_1(L_1R_2)$$
$$b_1(I_1^1)(R_1) = \sigma_1(R_1L_2) + \sigma_1(R_1R_2)$$
$$b_1(I_1^2)(L_2) = \sigma_1(L_2|R_1)$$
$$b_1(I_1^2)(R_2) = \sigma_1(R_2|R_1)$$



We call b_1 and σ_1 are equivalent.

More on Equivalent Strategies

The equivalence, by definition, holds at the leaf nodes too

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Claim

It is enough to check the equivalence only at the leaf nodes.

Reason: Pick an arbitrary non-leaf node, the probability of reaching that node is equal to the sum of the probabilites of reaching the leaf nodes in its subtree.

More on Equivalent Strategies

This argument can be extended further

Theorem (Utility Equivalence)

If σ_i and b_i are equivalent, then for every mixed/behavioral strategy vector of the other players ξ_{-i}, the following holds,

$$u_j(\sigma_i, \xi_{-i}) = u_j(b_i, \xi_{-i}), \ \forall j \in N.$$

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Repeat the argument for any equivalent mixed and behavioral strategy profiles.

Corollary

Let σ and b are equivalent, i.e., σ_i and b_i are equivalent $\forall i \in N$, then $u_i(\sigma) = u_i(b)$.

Contents

Equivalence of strategies in IIEFGs

Question

Why behavioral strategies are desirable?

Answer

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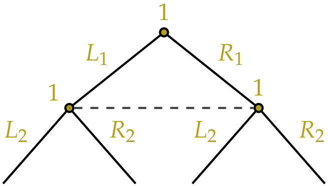
Question

Can we construct one from another?
OR
Does equivalence always hold?

Equivalence of strategies in IIEFGs (Example 1)

Player remembers that it made a move but forgets which move



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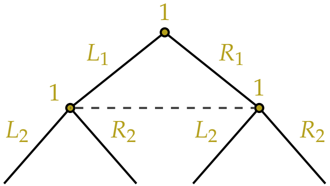

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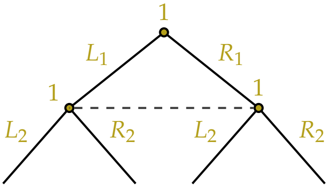

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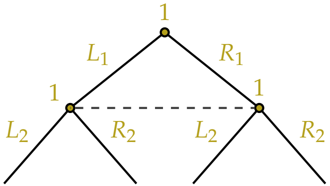

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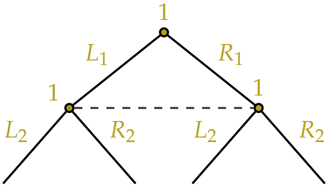

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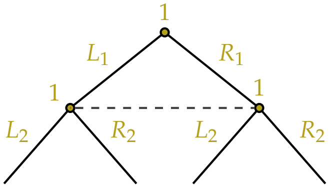

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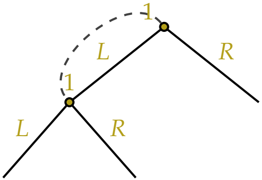

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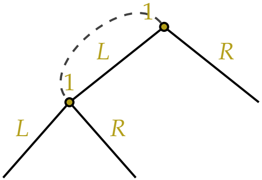

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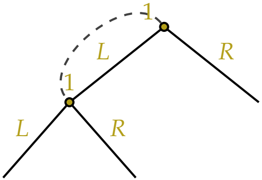

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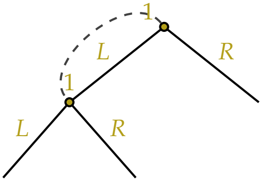

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Answer

The equivalence does not hold if the players are forgetful

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When does behavioral strategy have no equivalent mixed strategy?

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Observation from a graph viewpoint

1. Let x be a non-root node

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2. action at x_1 leading to x: the unique edge emanating from x_1 that is on the path from root to x

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4. If the path from the root to x passes through vertices x_1 and x_1' that are in the same information set of player i, and the action leading to x at x_1 and x_1' is different, then no **pure strategy** can ever lead to x

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5. Since mixed strategy is a randomization over pure strategies, every mixed strategy will put zero probability mass on x but behavioral strategy randomizes on every vertex **independently**, hence x may be reached in behavioral strategies with a positive probability

Equivalence of strategies in IIEFGs

The last observation can be stated as a lemma

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Lemma

If there exists a path from the root to some vertex x that passes through the same information set at least twice, and if the action leading to x is **not** *the same at each of those vertices, then the player at the information set has a behavioral strategy that has no equivalent mixed strategy.*

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The last observation can be stated as a lemma

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This lemma helps in proving the following characterization result of equivalence.

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Lemma

If there exists a path from the root to some vertex x that passes through the same information set at least twice, and if the action leading to x is **not** *the same at each of those vertices, then the player at the information set has a behavioral strategy that has no equivalent mixed strategy.*

This lemma helps in proving the following characterization result of equivalence.

Theorem (6.11 of MSZ)

Consider an IIEFG such that every vertex has at least two actions. Every behavioral strategy has an equivalent mixed strategy for a player iff each information set of that player intersects every path emanating from the root at most once.

Equivalence of strategies in IIEFGs

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Proof.

Homework. Reading exercise from MSZ. □

Contents

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To formalize (i.e., to set the conditions when the equivalence holds), we need to formalize the **forgetfulness** of a player.

- saw few examples of players' forgetfulness.
- our conditions need to ensure that none of the previous types of forgetfulness happens.

Behavioral Strategy equivalent to Mixed Strategy

Definition (Choice of **same action at an information set**)

Let $X = (x^0, x^1, \ldots, x^K)$ and $\hat{X} = (x^0, \hat{x}^1, \ldots, \hat{x}^L)$ be two paths in the game tree.

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We may use it for a path leading from a node x to some node y which is not an immediate child of x.

Games with Perfect Recall

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Rephrasing

For every I_i^j of player i and every pair of vertices $x, y \in I_i^j$, if the decision vertices of i are $x_i^1, x_i^2, \ldots, x_i^L = x$ and $y_i^1, y_i^2, \ldots, y_i^{L'} = y$ respectively for the two paths from the root to x and y, then

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3. $a_i(x_i^l \to x_i^{l+1}) = a_i(y_i^l \to y_i^{l+1}), \forall l = 1, 2, \ldots, L-1.$

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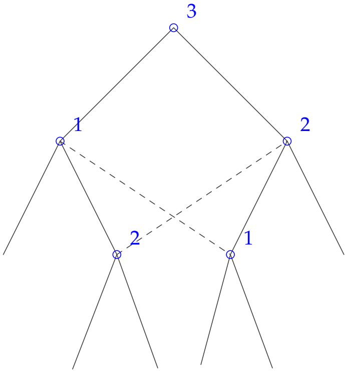
Note: Definition of perfect recall subsumes the condition where every behavioral strategy has equivalent mixed strategy

Example 1



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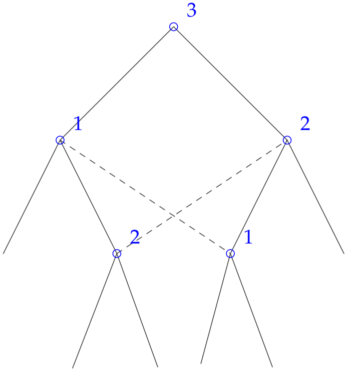

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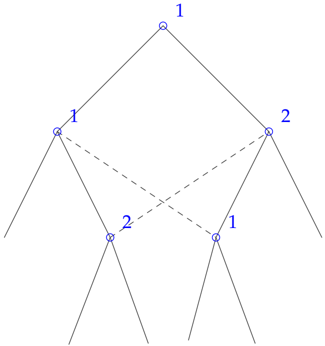
Game with Perfect Recall: This example satisfies the conditions of the definitions.

Example 2



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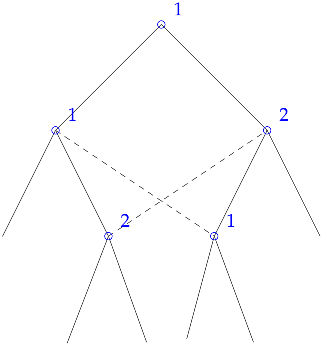

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Game with Imperfect Recall: Player 1 takes two different actions at the first information set to reach two different vertices of the second information set.

Implications of Perfect Recall

Let $S_i^*(x)$ be the set of pure strategies of player i at which he chooses actions leading to x, i.e., intersections of members of S_i with the path from root to x.

Theorem

If i is a player with perfect recall and x and x' are the two vertices in the same information set of i, then $S_i^*(x) = S_i^*(x')$.

The above conclusion comes from the same sequence of information sets and same actions. The next implication of mixed and behavioral strategies.

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Theorem (Kuhn 1957)

In every IIEFG, if i is a player with perfect recall, then for every mixed strategy of i, there exists a behavioral strategy.

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- Proof left as reading exercise (MSZ Theorem 6.15)
- The proof is constructive. It starts with the mixed strategy and constructs the behavioral strategies such that the probabilities of reaching a leaf are same. The arguments show that such a construction is always possible because of perfect recall.

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