

भारतीय प्रौद्योगिकी संस्थान मुंबई

Indian Institute of Technology Bombay

# CS 6001: Game Theory and Algorithmic Mechanism Design

Week 6

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Slide preparation acknowledgments: Ramsundar Anandanarayanan and Harshvardhan Agarwal

ज्ञानम् परमम् ध्येयम् Knowledge is the supreme goal

### Contents



- ► Equilibrium in IIEFGs
- ▶ Game Theory in Practice: P2P File Sharing
- ► Bayesian Games
- ▶ Strategy, Utility in Bayesian Games
- Equilibrium in Bayesian Games
- ▶ Examples in Bayesian Equilibrium



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### Belief

It is the conditional probability distribution over the histories in an information set - conditioned on reaching the information set.







Consider the behavioral strategy profile:  $\sigma_1$ , at  $I_1^1(L\{5/12\}, M\{4/12\}, R\{3/12\})$ 



Consider the behavioral strategy profile:  $\sigma_2$ , at  $I_2^1(l\{1\}, m\{0\}, r\{0\})$  choose l



Consider the behavioral strategy profile:  $\sigma_1$ , at  $I_1^2(L_1\{0\}, R_1\{1\})$  choose  $R_1$ 



Consider the behavioral strategy profile:  $\sigma_1$ , at  $I_1^3(L_2\{1\}, R_2\{0\})$  choose  $L_2$ 





#### Question

Is this an equilibrium? which implies

- Are the Bayesian beliefs consistent with  $P_{\sigma}$  that visits vertex *x* with probability  $P_{\sigma}(x)$ ?
- The actions and beliefs are consistent for every player, i.e., maximizes their expected utility?





#### Sequential rationality

Choose an action maximizing expected utility at each information set.

The strategy vector  $\sigma$  induces the following probabilities to the vertices.  $P_{\sigma}(x_2) = 5/12, P_{\sigma}(x_3) = 4/12, P_{\sigma}(x_4) = 0, P_{\sigma}(x_5) = 0, P_{\sigma}(x_6) = 4/12, P_{\sigma}(x_7) = 0$ 





- Player 1 at information set  $I_1^3$ , believes that  $x_6$  is reached with probability 1.
- If the belief was > 2/7 in favor of  $x_7$ , player 1 should have chosen  $R_2$





- Player 2 at  $I_2^1$  believes the  $x_3$  is reached w.p.  $P_{\sigma}(x_3|I_2^1) = P_{\sigma}(x_3)/(P_{\sigma}(x_2) + P_{\sigma}(x_3)) = 4/9$
- Similarly  $P_{\sigma}(x_2|I_2^1) = 5/9$





#### Question

Is the action of player 2 sequentially rational w.r.t. her belief?

#### Answer

By picking *l*, expected utility =  $5/9 \times 1 + 4/9 \times 2 = 13/9$ , larger than any other choice of action.





#### Question

Given all information, what is the sequentially rational strategy for player 1 at  $I_1^1$ 

#### Answer

L, M, R all give the same expected utility for player 1 (utility = 2).



Thus, mixed/behavioral strategy profile  $\sigma$  is sequentially rational for all players.



# Belief Let the information sets of player *i* be $I_i = \{I_i^1, I_i^2, I_i^3, ..., I_i^{k(i)}\}$ . The belief of player *i* is a mapping $\mu_i^j : I_i^j \to [0, 1]$ s.t., $\sum_{x \in I_i^j} \mu_i^j(x) = 1$



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#### **Bayesian Belief**

A **belief**  $\mu_i = {\mu_i^1, \mu_i^2, ..., \mu_i^{k(i)}}$  of player *i* is **Bayesian** w.r.t. to the behavioral strategy  $\sigma$ , if it is derived from  $\sigma$  using Bayes rule, i.e.,

$$\mu_{i}^{j}(x) = P_{\sigma}(x) / \sum_{y \in I_{i}^{j}} P_{\sigma}(y), \forall x \in I_{i}^{j}, \forall j = 1, 2, 3, ..., k(i)$$



A strategy  $\sigma_i$  of player i at an information set  $l_i^j$  is **sequentially rational** given  $\sigma_{-i}$  and partial belief  $\mu_i^j$  if

$$\sum_{x \in I_i^j} \mu_i^j(x) u_i(\sigma_i, \sigma_{-i} | x) \ge \sum_{x \in I_i^j} \mu_i^j(x) u_i(\sigma_i', \sigma_{-i} | x)$$



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- Sequential rationality is a refinement of Nash Equilibrium.
- The notion coincides with SPNE when applied to PIEFGs



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Equilibrium with Sequential Rationality

Perfect Bayesian Equilibrium: An assessment ( $\sigma$ ,  $\mu$ ) is PBE if  $\forall i \in N$ 

- $\mu_i$  is Bayesian w.r.t.  $\sigma$
- $\sigma_i$  is sequentially rational given  $\sigma_{-i}$  and  $\mu_i$



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- Often represented only with  $\sigma$ , since  $\mu$  is obtained from  $\sigma$
- Self-enforcing (like the SPNE) in a Bayesian way.





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**Peer to Peer**<sup>1</sup>





<sup>1</sup>Slides of this section are adapted from CS186, Harvard



• Scalability

**Terminology:** 



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- Failure resilience

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• **Protocol:** messages that can be sent, actions that can be taken over the network



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- Client: a particular process for sending messages, taking actions
# **Desired Properties and Terminology**



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- Failure resilience

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- **Reference client:** particular implementation

# **Desired Properties and Terminology**



- Scalability
- Failure resilience

**Terminology:** 

- **Protocol:** messages that can be sent, actions that can be taken over the network
- Client: a particular process for sending messages, taking actions
- **Reference client:** particular implementation
- Peer



Napster (1999 - 2001)

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```
Gnutella (2000 - )
```

- Get list of IP addresses of peers from set of known peers (no server)
- To get a file: Query message broadcast by peer A to known peers
- Query response: sent by B if B has the desired file (routed back to requestor)
- A can then download directly from B

## The File Sharing Game





(Gnutella) File Sharing Game



#### Rank Ordering of Peers by Number of Files Shared



Image courtesy: Adar and Huberman (2000)



- Client developers can ensure file sharing
- But competition among the developers



- Client developers can ensure file sharing
- But competition among the developers
- 85% peers free-riding by 2005; Gnutella less than 1% of ww P2P traffic by 2013
- Few other P2P systems met the same fate



**BitTorrent** (2001 - )

- Approx 85% of P2P traffic in US
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Key innovations

- Break file into pieces: A repeated game!
- "If you let me download, I'll reciprocate."

## **BitTorrent Schematic**





Figure 5.4.: Starting a download process in the BitTorrent protocol: 1) A user goes to a searchable directory to find a link to a .torrent file corresponding to the desired content; 2) the .torrent file contains metadata about the content, in particular the URL of a tracker; 3) the tracker provides a list of peers participating in the swarm for the content (i.e., their IP address and port); 4) the user's BitTorrent client can now contact all these peers and download content.

Image courtesy: Parkes and Seuken (2017)





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Strategy of the seeder is tit-for-tat

# Illustration



Illustration



- How often to contact tracker?
- Which pieces to reveal?
- How many upload slots, which peers to unchoke, at what speed?
- What data to allow others to download?
- Possible goals: min *upload*, *maxdownloadspeed*, *somebalance*

## **Attacks on BitTorrent**



- BitThief
- Strategic piece revealer
- BitTyrant



- Goal: download files without uploading
- Keep asking for peers from tracker, grow neighborhood quickly
- Exploit the optimistic unchoking part
- Never upload!



- Goal: download files without uploading
- Keep asking for peers from tracker, grow neighborhood quickly
- Exploit the optimistic unchoking part
- Never upload!
- Fix: modify the tracker (block same IP address within 30).

Ref: Locher et al., "Free Riding in BitTorrent is Cheap", HotNets 2006



- Reference client: tell neighbors about new pieces, use "rarest-first" to request
- Manipulator strategy: reveal most common piece that reciprocating peer does not have!
- Try to protect a monopoly, keep others interested

Ref: Levin et al., "BitTorrent is an Auction: Analyzing and Improving BitTorrent's Incentives", SIGCOMM 2008

## **Strategic Piece Revealer**









• P2P demonstrates importance of game-theory in computer systems





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- Early systems were easily manipulated
- BitTorrent's innovation was to break files into pieces, enabling TitForTat.
- Still some vulnerabilities, but generally very successful example of incentive-based protocol design.





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Games

• Non-cooperative games



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- Complete information - Players deterministically know which game they are playing



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- Cooperative games Players form coalitions and utilities are defined over coalitions
- Other types of games repeated, stochastic etc.

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Games with Incomplete information

- Players do not know deterministically know which game they are playing
- They receive **private signals / types**
- To discuss: a special subclass called games with incomplete information with **common priors** (Harsanyi 1967)
- Also called **Bayesian games**



## **Bayesian Games: Example**





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- $\Gamma_{\theta}$ : NFG for the type profile  $\theta \in \Theta$  i.e.,  $\Gamma_{\theta} = \langle N, (A_i(\theta))_{i \in N}, (u_i(\theta))_{i \in N} \rangle$  $u_i : A \times \Theta \to \mathbb{R}, A = \times_{i \in N} A_i$  [We assume  $A_i(\theta) = A_i, \forall \theta$ ]





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- Player *i* realizes a payoff of  $u_i(a_i, a_{-i}; \theta_i, \theta_{-i})$





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- Ex-ante utility
- Ex-interim utility
- Ex-post utility (for complete information game)

## **Ex-ante Utility**



#### Definition (Ex-ante utility)

Expected utility **before** observing own type.

$$u_i(\sigma) = \sum_{\theta \in \Theta} \frac{P(\theta)u_i(\sigma(\theta); \theta)}{\sum_{\substack{\theta \in \Theta}} P(\theta) \sum_{(a_1, a_2, \dots, a_n) \in A} \prod_{j \in N} \sigma_j(\theta_j)[a_j]u_i(a_1, \dots, a_n; \theta_1, \dots, \theta_n)}$$

## **Ex-ante Utility**



#### Definition (Ex-ante utility)

Expected utility **before** observing own type.

$$u_i(\sigma) = \sum_{\theta \in \Theta} \frac{P(\theta)u_i(\sigma(\theta); \theta)}{\sum_{\substack{\theta \in \Theta}} P(\theta) \sum_{(a_1, a_2, \dots, a_n) \in A} \prod_{j \in N} \sigma_j(\theta_j)[a_j]u_i(a_1, \dots, a_n; \theta_1, \dots, \theta_n)}$$

The **belief** of player *i* over others' types changes after observing her own type  $\theta_i$ :

$$P(\theta_{-i}|\theta_i) = \frac{P(\theta_i, \theta_{-i})}{\sum_{\tilde{\theta}_{-i} \in \Theta_{-i}} P(\theta_i, \tilde{\theta}_{-i})} \qquad \text{Bayes rule}$$



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This is why we needed every marginal to be positive



Definition (Ex-interim utility)

Expected utility after observing one's own type.

$$u_i(\sigma|\theta_i) = \sum_{\theta_{-i}\in\Theta_{-i}} P(\theta_{-i}|\theta_i) u_i(\sigma(\theta);\theta)$$



Definition (Ex-interim utility)

Expected utility after observing one's own type.

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**Special Case:** for independent types, observing  $\theta_i$  does not give any information on  $\theta_{-i}$ 



Definition (Ex-interim utility)

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$$u_i(\sigma|\theta_i) = \sum_{\theta_{-i}\in\Theta_{-i}} \frac{P(\theta_{-i}|\theta_i)u_i(\sigma(\theta);\theta)}{\Phi_i(\sigma(\theta);\theta)}$$

**Special Case:** for independent types, observing  $\theta_i$  does not give any information on  $\theta_{-i}$ 

Relation between the two utilities is given by

$$u_i(\sigma) = \sum_{\theta_i \in \Theta_i} P(\theta_i) u_i(\sigma | \theta_i)$$

# Example 1: Two Player Bargaining Game



• Player 1 : seller, type : price at which he is willing to sell

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- Player 1 : seller, type : price at which he is willing to sell
- Player 2 : buyer, type : price at which he is willing to buy
- $\Theta_1 = \Theta_2 = \{1, 2, \dots, 100\}, A_1 = A_2 = \{1, 2, \dots, 100\}$
- If the bid of the seller is smaller or equal to that of the buyer, trade happens at a price average of the two bids. Else, trade does not happen.

Suppose type generation is independent and uniform over  $\Theta_1, \Theta_2$  respectively,

$$P(\theta_2|\theta_1) = P(\theta_2) = \frac{1}{100}, \forall \theta_1, \theta_2$$
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$$u_{1}(a_{1}, a_{2}; \theta_{1}, \theta_{2}) = \begin{cases} \frac{a_{1}+a_{2}}{2} - \theta_{1} & \text{if } a_{2} \ge a_{1} \\ 0 & \text{otherwise} \end{cases}$$
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Common Prior :  $P(\theta_1, \theta_2) = \frac{1}{10000}, \forall \theta_1, \theta_2$ 



**Allocation Function:** 

$$O_1(b_1, b_2) = \begin{cases} 1 & \text{if } b_1 \ge b_2 \\ 0 & \text{ow} \end{cases} \qquad O_2(b_1, b_2) = \begin{cases} 1 & \text{if } b_2 > b_1 \\ 0 & \text{ow} \end{cases}$$

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**Beliefs:** 

$$\begin{split} f(\theta_2|\theta_1) &= 1, \forall \theta_1, \theta_2 \\ f(\theta_1|\theta_2) &= 1, \forall \theta_1, \theta_2 \\ f(\theta_1, \theta_2) &= 1, \forall \theta_1, \theta_2 \end{split}$$

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$$u_i(b_1, b_2; \theta_1, \theta_2) = O_i(b_1, b_2)(\theta_i - b_i)$$

Winner pays his/her bid.





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# Equilibrium concepts in Bayesian games



Ex-ante: before observing her own type

**Nash Equilibrium**  $(\sigma^*, P)$ :  $u_i(\sigma^*_i, \sigma^*_{-i}) \ge u_i(\sigma'_i, \sigma^*_{-i}), \forall \sigma'_i, \forall i \in N$ 



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Ex-interim: after observing her own type

**Bayesian Equilibrium**  $(\sigma^*, P)$ :  $u_i(\sigma_i^*(\theta_i), \sigma_{-i}^*|\theta_i) \ge u_i(\sigma_i'(\theta_i), \sigma_{-i}^*|\theta_i), \forall \sigma_i', \forall \theta_i \in \Theta_i, \forall i \in N$ 

$$u_i(\sigma) = \sum_{\theta \in \Theta} P(\theta) u_i(\sigma(\theta); \theta), \qquad u_i(\sigma|\theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} P(\theta_{-i}|\theta_i) u_i(\sigma(\theta); \theta)$$



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$$u_i(\sigma) = \sum_{\theta \in \Theta} P(\theta) u_i(\sigma(\theta); \theta), \qquad u_i(\sigma|\theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} P(\theta_{-i}|\theta_i) u_i(\sigma(\theta); \theta)$$

• The RHS of the definition can be replaced by a pure strategy *a<sub>i</sub>*, ∀*a<sub>i</sub>* ∈ *A<sub>i</sub>*. The reason is exactly the same as that of MSNE (these definitions are equivalent)



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- The RHS of the definition can be replaced by a pure strategy *a<sub>i</sub>*, *∀a<sub>i</sub>* ∈ *A<sub>i</sub>*. The reason is exactly the same as that of MSNE (these definitions are equivalent)
- NE takes expectation over  $P(\theta)$



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$$u_i(\sigma) = \sum_{\theta \in \Theta} P(\theta) u_i(\sigma(\theta); \theta), \qquad u_i(\sigma|\theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} P(\theta_{-i}|\theta_i) u_i(\sigma(\theta); \theta)$$

- The RHS of the definition can be replaced by a pure strategy *a<sub>i</sub>*, *∀a<sub>i</sub>* ∈ *A<sub>i</sub>*. The reason is exactly the same as that of MSNE (these definitions are equivalent)
- NE takes expectation over  $P(\theta)$
- BE takes expectation over  $P(\theta_{-i}|\theta_i)$



In finite Bayesian games, a strategy profile is Bayesian Equilibrium iff it is a Nash equilibrium



In finite Bayesian games, a strategy profile is Bayesian Equilibrium iff it is a Nash equilibrium

## Proof.

For the forward direction, suppose  $(\sigma^*, P)$  is a Bayesian equilibrium, consider

$$\begin{aligned} u_i(\sigma'_i, \sigma^*_{-i}) &= \sum_{\theta_i \in \Theta_i} P(\theta_i) u_i(\sigma'_i(\theta_i), \sigma^*_{-i} | \theta_i) \\ &\leqslant \sum_{\theta_i \in \Theta_i} P(\theta_i) u_i(\sigma^*_i(\theta_i), \sigma^*_{-i} | \theta_i), \text{ since } (\sigma^*, P) \text{ is a BE} \\ &= u_i(\sigma^*_i, \sigma^*_{-i}) \end{aligned}$$



In finite Bayesian games, a strategy profile is Bayesian Equilibrium iff it is a Nash equilibrium



In finite Bayesian games, a strategy profile is Bayesian Equilibrium iff it is a Nash equilibrium

## Proof.

For the reverse direction, proof by contradiction. Suppose  $(\sigma^*, P)$  is not a Bayesian equilibrium i.e., there exists some  $i \in N$ , some  $\theta_i \in \Theta_i$ , some  $a_i \in A_i$ , s.t.

 $u_i(a_i, \sigma^*_{-i}|\theta_i) > u_i(\sigma^*_i(\theta_i), \sigma^*_{-i}|\theta_i)$ 



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 $u_i(a_i, \sigma_{-i}^*|\theta_i) > u_i(\sigma_i^*(\theta_i), \sigma_{-i}^*|\theta_i)$ 

Construct the strategy  $\hat{\sigma}_i$  s.t.,

$$\hat{\sigma_i}( heta_i') = \sigma_i^*( heta_i'), orall heta_i' \in \Theta_i \setminus \{ heta_i\}$$



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$$\hat{\sigma}_i( heta_i') = \sigma_i^*( heta_i'), orall heta_i' \in \Theta_i \setminus \{ heta_i\}$$

 $\hat{\sigma}_i(\theta_i)[a_i] = 1, \hat{\sigma}_i(\theta_i)[b_i] = 0, \forall b_i \in A_i \setminus \{a_i\}$ 



In finite Bayesian games, a strategy profile is Bayesian Equilibrium iff it is a Nash equilibrium

Proof.



In finite Bayesian games, a strategy profile is Bayesian Equilibrium iff it is a Nash equilibrium

## Proof.

$$u_i(\hat{\sigma}_i, \sigma_{-i}^*) = \sum_{\tilde{\theta}_i \in \Theta_i} P(\tilde{\theta}_i) u_i(\hat{\sigma}_i(\tilde{\theta}_i), \sigma_{-i}^* | \tilde{\theta}_i)$$



In finite Bayesian games, a strategy profile is Bayesian Equilibrium iff it is a Nash equilibrium

## Proof.

$$u_{i}(\hat{\sigma}_{i}, \sigma_{-i}^{*}) = \sum_{\tilde{\theta}_{i} \in \Theta_{i}} P(\tilde{\theta}_{i}) u_{i}(\hat{\sigma}_{i}(\tilde{\theta}_{i}), \sigma_{-i}^{*}|\tilde{\theta}_{i})$$
$$= \sum_{\tilde{\theta}_{i} \in \Theta_{i} \setminus \{\theta_{i}\}} P(\tilde{\theta}_{i}) u_{i}(\hat{\sigma}_{i}(\tilde{\theta}_{i}), \sigma_{-i}^{*}|\tilde{\theta}_{i})$$



In finite Bayesian games, a strategy profile is Bayesian Equilibrium iff it is a Nash equilibrium

## Proof.

$$u_{i}(\hat{\sigma}_{i}, \sigma_{-i}^{*}) = \sum_{\tilde{\theta}_{i} \in \Theta_{i}} P(\tilde{\theta}_{i}) u_{i}(\hat{\sigma}_{i}(\tilde{\theta}_{i}), \sigma_{-i}^{*} | \tilde{\theta}_{i})$$
  
$$= \sum_{\tilde{\theta}_{i} \in \Theta_{i} \setminus \{\theta_{i}\}} P(\tilde{\theta}_{i}) u_{i}(\hat{\sigma}_{i}(\tilde{\theta}_{i}), \sigma_{-i}^{*} | \tilde{\theta}_{i}) + P(\theta_{i}) u_{i}(\hat{\sigma}_{i}(\theta_{i}), \sigma_{-i}^{*} | \theta_{i})$$



In finite Bayesian games, a strategy profile is Bayesian Equilibrium iff it is a Nash equilibrium

## Proof.

$$u_{i}(\hat{\sigma}_{i}, \sigma_{-i}^{*}) = \sum_{\tilde{\theta}_{i} \in \Theta_{i}} P(\tilde{\theta}_{i}) u_{i}(\hat{\sigma}_{i}(\tilde{\theta}_{i}), \sigma_{-i}^{*} | \tilde{\theta}_{i})$$

$$= \sum_{\tilde{\theta}_{i} \in \Theta_{i} \setminus \{\theta_{i}\}} P(\tilde{\theta}_{i}) u_{i}(\hat{\sigma}_{i}(\tilde{\theta}_{i}), \sigma_{-i}^{*} | \tilde{\theta}_{i}) + P(\theta_{i}) u_{i}(\hat{\sigma}_{i}(\theta_{i}), \sigma_{-i}^{*} | \theta_{i})$$

$$> \sum_{\tilde{\theta}_{i} \in \Theta_{i} \setminus \{\theta_{i}\}} P(\tilde{\theta}_{i}) u_{i}(\sigma_{i}^{*}(\tilde{\theta}_{i}), \sigma_{-i}^{*} | \tilde{\theta}_{i}) + P(\theta_{i}) u_{i}(\sigma_{i}^{*}(\theta_{i}), \sigma_{-i}^{*} | \theta_{i}) = u_{i}(\sigma_{i}^{*}, \sigma_{-i}^{*})$$



In finite Bayesian games, a strategy profile is Bayesian Equilibrium iff it is a Nash equilibrium

## Proof.

Reverse direction proof continued ...

$$\begin{split} u_{i}(\hat{\sigma}_{i},\sigma_{-i}^{*}) &= \sum_{\tilde{\theta}_{i}\in\Theta_{i}} P(\tilde{\theta}_{i})u_{i}(\hat{\sigma}_{i}(\tilde{\theta}_{i}),\sigma_{-i}^{*}|\tilde{\theta}_{i}) \\ &= \sum_{\tilde{\theta}_{i}\in\Theta_{i}\setminus\{\theta_{i}\}} P(\tilde{\theta}_{i})u_{i}(\hat{\sigma}_{i}(\tilde{\theta}_{i}),\sigma_{-i}^{*}|\tilde{\theta}_{i}) + P(\theta_{i})u_{i}(\hat{\sigma}_{i}(\theta_{i}),\sigma_{-i}^{*}|\theta_{i}) \\ &> \sum_{\tilde{\theta}_{i}\in\Theta_{i}\setminus\{\theta_{i}\}} P(\tilde{\theta}_{i})u_{i}(\sigma_{i}^{*}(\tilde{\theta}_{i}),\sigma_{-i}^{*}|\tilde{\theta}_{i}) + P(\theta_{i})u_{i}(\sigma_{i}^{*}(\theta_{i}),\sigma_{-i}^{*}|\theta_{i}) = u_{i}(\sigma_{i}^{*},\sigma_{-i}^{*}) \end{split}$$

Hence,  $(\sigma_i^*, \sigma_{-i}^*)$  is not a Nash equilibrium



*Every finite Bayesian game has a Bayesian equilibrium.* 

[Finite Bayesian game: set of players, action set and type set are finite]



*Every finite Bayesian game has a Bayesian equilibrium.* 

[Finite Bayesian game: set of players, action set and type set are finite]

#### Proof.

Proof idea: Transform the Bayesian game into a complete information game treating each type as a player, and invoke Nash Theorem for the existence of equilibrium - which is a BE in the original game. [See addendum for details]





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**Allocation Function** 

$$O_1(b_1, b_2) = \mathbb{1}\{b_1 \ge b_2\}$$
$$O_2(b_1, b_2) = \mathbb{1}\{b_2 > b_1\}$$

Beliefs

$$f(\theta_2|\theta_1) = 1, \forall \theta_1, \theta_2$$
  

$$f(\theta_1|\theta_2) = 1, \forall \theta_1, \theta_2$$
  

$$f(\theta_1, \theta_2) = 1, \forall \theta_1, \theta_2$$



• If  $b_1 \ge b_2$ , player 1 wins and pays her bid; otherwise, player 2 wins and pays her bid.

$$u_1(b_1, b_2, \theta_1, \theta_2) = (\theta_1 - b_1) \mathbb{1} \{ b_1 \ge b_2 \}$$
  
$$u_2(b_1, b_2, \theta_1, \theta_2) = (\theta_2 - b_2) \mathbb{1} \{ b_1 < b_2 \}$$



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$$u_2(b_1, b_2, \theta_1, \theta_2) = (\theta_2 - b_2) \mathbb{1}\{b_1 < b_2\}$$

•  $b_1 = s_1(\theta_1), b_2 = s_2(\theta_2)$ Assume  $s_i(\theta_i) = \alpha_i \theta_i, \alpha_i > 0, i = 1, 2$ 



## To find the BE, we need to find the $s_i^*$ (or $\alpha_i^*$ ) that maximizes the ex-interim utility of player *i*. i.e.

 $\max_{\sigma_i} u_i(\sigma_i, \sigma_{-i}^* | \theta_i)$ 

For player 1, this reduces to

$$\max_{\sigma_i} u_i(\sigma_i, \sigma_{-i}^* | \theta_i) = \max_{b_1 \in [0,1]} \int_0^1 f(\theta_2 | \theta_1) (\theta_1 - b_1) \mathbb{1}\{b_1 \ge \alpha_2 \theta_2\} d\theta_2$$
$$= \max_{b_1 \in [0,1]} (\theta_1 - b_1) \frac{b_1}{\alpha_2}$$
$$\implies b_1 = \frac{\theta_1}{2}$$



From this we get,

$$s_1^*(\theta_1) = \frac{\theta_1}{2}$$
$$s_2^*(\theta_2) = \frac{\theta_2}{2}$$

is a BE.

In the Bayesian Game induced by uniform prior on first price auction, bidding half the true value is a Bayesian equilibrium.



Highest bidder wins but pays the second highest bid.

$$u_1(b_1, b_2, \theta_1, \theta_2) = (\theta_1 - b_2) \mathbb{1}\{b_1 \ge b_2\}$$
$$u_2(b_1, b_2, \theta_1, \theta_2) = (\theta_2 - b_1) \mathbb{1}\{b_1 < b_2\}$$



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$$u_2(b_1, b_2, \theta_1, \theta_2) = (\theta_2 - b_1) \mathbb{1}\{b_1 < b_2\}$$

Player 1 has to maximize

$$= \int_{0}^{1} f(\theta_{2}|\theta_{1})(\theta_{1} - s_{2}(\theta_{2}))\mathbb{1}\{b_{1} \ge s_{2}(\theta_{2})\}d\theta_{2}$$
  
$$= \int_{0}^{1} \mathbb{1} \cdot (\theta_{1} - \alpha_{2}\theta_{2})\mathbb{1}\{\theta_{2} \le \frac{b_{1}}{\alpha_{2}}\}d\theta_{2}$$
  
$$= \frac{1}{\alpha_{2}}(b_{1}\theta_{1} - \frac{\theta_{1}^{2}}{2})$$

This is maximized when  $b_1 = \theta_1$ . Similarly for  $b_2 = \theta_2$ .


## If the distribution of $\theta_1$ and $\theta_2$ were arbitrary but independent, the maximization problem would have been

$$\int_0^{\frac{b_1}{\alpha_2}} f(\theta_2)(\theta_1 - \alpha_2\theta_2)d\theta_2 = \theta_1 F\left(\frac{b_1}{\alpha_2}\right) - \alpha_2 \int_0^{\frac{b_1}{\alpha_2}} \theta_2 f(\theta_2)d\theta_2$$



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Differentiating w.r.t.  $b_1$ , we get

$$\theta_{1} \frac{1}{\alpha_{2}} f\left(\frac{b_{1}}{\alpha_{2}}\right) - \alpha_{2} \cdot \frac{b_{1}}{\alpha_{2}} f\left(\frac{b_{1}}{\alpha_{2}}\right) \frac{1}{\alpha_{2}} = 0 \implies f\left(\frac{b_{1}}{\alpha_{2}}\right) (b_{1} - \theta_{1}) = 0 \tag{1}$$
$$\implies b_{1} = \theta_{1}, \text{ if } f\left(\frac{b_{1}}{\alpha_{2}}\right) > 0 \tag{2}$$

Similarly for player 2.



## If the distribution of $\theta_1$ and $\theta_2$ were arbitrary but independent, the maximization problem would have been

$$\int_0^{\frac{b_1}{\alpha_2}} f(\theta_2)(\theta_1 - \alpha_2 \theta_2) d\theta_2 = \theta_1 F\left(\frac{b_1}{\alpha_2}\right) - \alpha_2 \int_0^{\frac{b_1}{\alpha_2}} \theta_2 f(\theta_2) d\theta_2$$

Differentiating w.r.t.  $b_1$ , we get

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$$\implies b_{1} = \theta_{1}, \text{ if } f\left(\frac{b_{1}}{\alpha_{2}}\right) > 0 \tag{2}$$

Similarly for player 2. For any independent positive prior, bidding true type is a BE of the induced Bayesian game in Second Price Auction.



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