



भारतीय प्रौद्योगिकी संस्थान मुंबई  
Indian Institute of Technology Bombay

# CS 6001: Game Theory and Algorithmic Mechanism Design

Week 6

Swaprava Nath

Slide preparation acknowledgments: Ramsundar Anandanarayanan and Harshvardhan Agarwal

ज्ञानम् परमम् ध्येयम्

Knowledge is the supreme goal



- ▶ Equilibrium in IIEFGs
- ▶ Game Theory in Practice: P2P File Sharing
- ▶ Bayesian Games
- ▶ Strategy, Utility in Bayesian Games
- ▶ Equilibrium in Bayesian Games
- ▶ Examples in Bayesian Equilibrium



- Can extend the subgame perfection of PIEFG, but since the nodes/histories are uncertain, we need to extend to mixed strategies



- Can extend the subgame perfection of PIEFG, but since the nodes/histories are uncertain, we need to extend to mixed strategies
- Because of the information sets, best response cannot be defined without the belief of each player



- Can extend the subgame perfection of PIEFG, but since the nodes/histories are uncertain, we need to extend to mixed strategies
- Because of the information sets, best response cannot be defined without the belief of each player

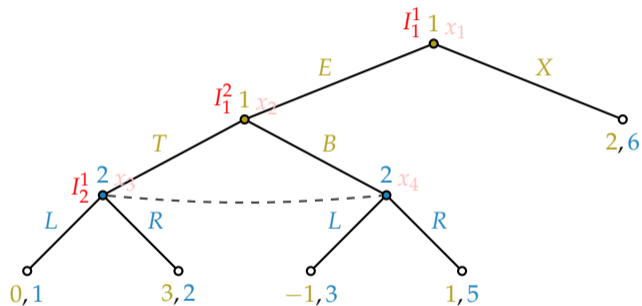


- Can extend the subgame perfection of PIEFG, but since the nodes/histories are uncertain, we need to extend to mixed strategies
- Because of the information sets, best response cannot be defined without the belief of each player

## Belief

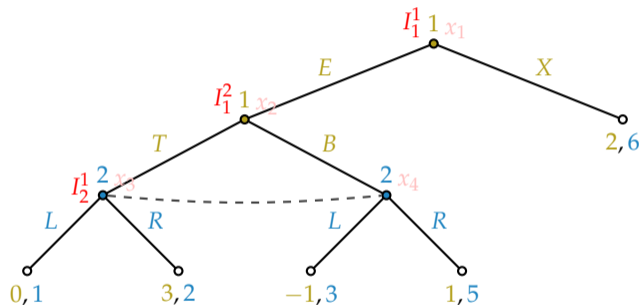
It is the conditional probability distribution over the histories in an information set - conditioned on reaching the information set.

# Example 1: IIEFG and Subgame Perfection



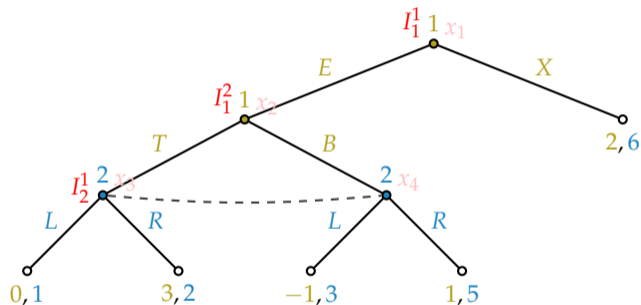


# Example 1: IIEFG and Subgame Perfection



- Two subgames: rooted at  $x_1$  and  $x_2$

# Example 1: IIEFG and Subgame Perfection

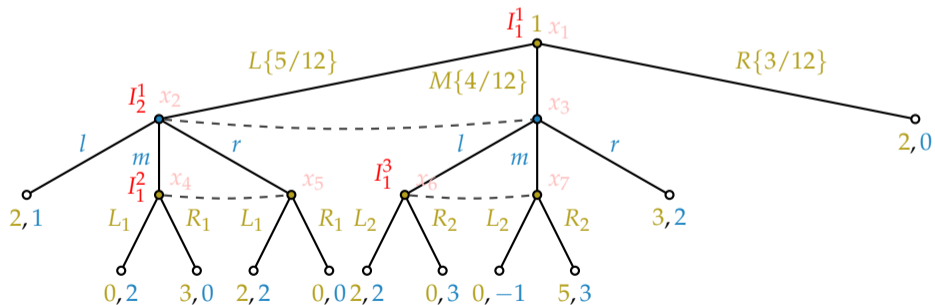


- Two subgames: rooted at  $x_1$  and  $x_2$
- Backward induction: solve at  $x_2$ :  $R$  dominates  $L$  for player 2, then  $T$  dominates  $B$  for player 1



## Example 2: IIEFG and Sequential Rationality

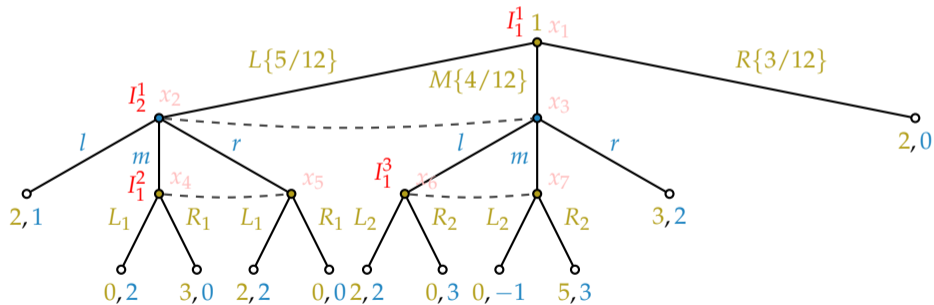
Example 7.38 MSZ: An IIEFG with perfect recall, i.e., mixed and behavioral strategies are equivalent.





## Example 2: IIEFG and Sequential Rationality

Example 7.38 MSZ: An IIEFG with perfect recall, i.e., mixed and behavioral strategies are equivalent.

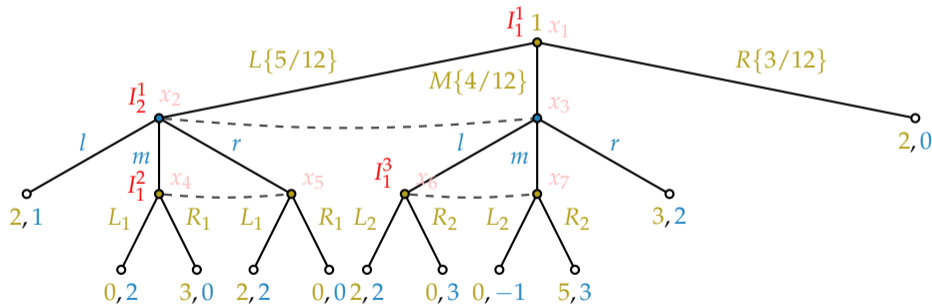


Consider the behavioral strategy profile:  $\sigma_1$ , at  $I_1^1(L\{5/12\}, M\{4/12\}, R\{3/12\})$



## Example 2: IIEFG and Sequential Rationality

Example 7.38 MSZ: An IIEFG with perfect recall, i.e., mixed and behavioral strategies are equivalent.

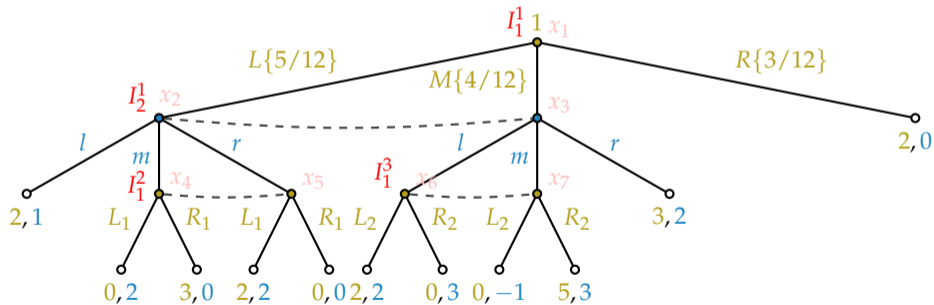


Consider the behavioral strategy profile:  $\sigma_2$ , at  $I_2^1(l\{1\}, m\{0\}, r\{0\})$  choose  $l$



## Example 2: IIEFG and Sequential Rationality

Example 7.38 MSZ: An IIEFG with perfect recall, i.e., mixed and behavioral strategies are equivalent.

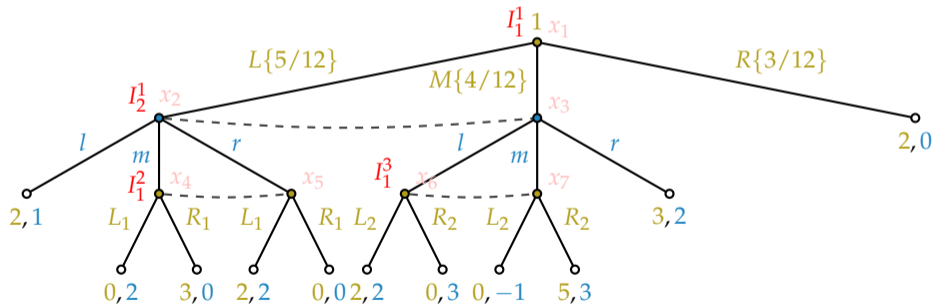


Consider the behavioral strategy profile:  $\sigma_1$ , at  $I_1^2(L_1\{0\}, R_1\{1\})$  choose  $R_1$



## Example 2: IIEFG and Sequential Rationality

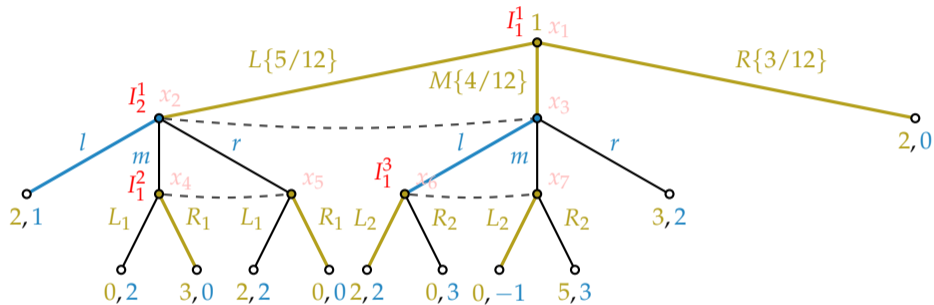
Example 7.38 MSZ: An IIEFG with perfect recall, i.e., mixed and behavioral strategies are equivalent.



Consider the behavioral strategy profile:  $\sigma_1$ , at  $I_1^3(L_2\{1\}, R_2\{0\})$  choose  $L_2$



# Example: An IIEFG with perfect recall



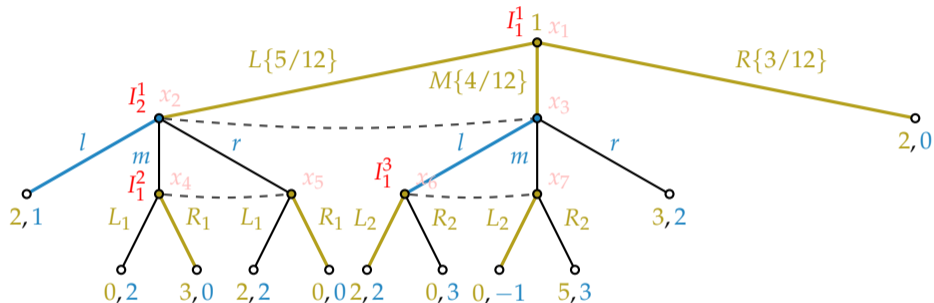
## Question

Is this an equilibrium?  
which implies

- Are the Bayesian beliefs consistent with  $P_\sigma$  - that visits vertex  $x$  with probability  $P_\sigma(x)$ ?
- The actions and beliefs are consistent for every player, i.e., maximizes their expected utility?



# Example: An IIEFG with perfect recall



## Sequential Rationality

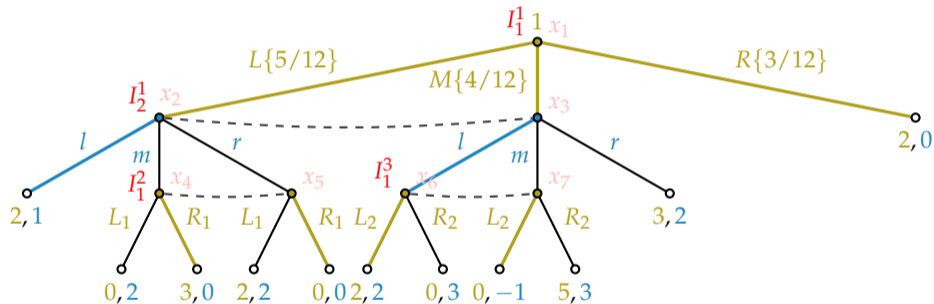
Choose an action maximizing expected utility at each information set.

The strategy vector  $\sigma$  induces the following probabilities to the vertices.

$$P_\sigma(x_2) = 5/12, P_\sigma(x_3) = 4/12, P_\sigma(x_4) = 0, P_\sigma(x_5) = 0, P_\sigma(x_6) = 4/12, P_\sigma(x_7) = 0$$



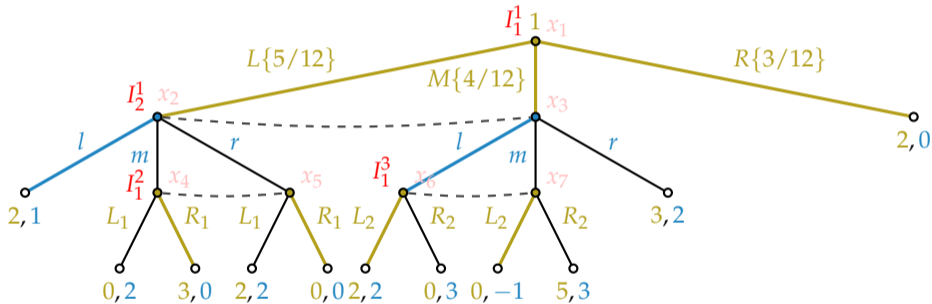
# Example: An IIEFG with perfect recall



- Player 1 at information set  $I_1^3$ , believes that  $x_6$  is reached with probability 1.
- If the belief was  $> 2/7$  in favor of  $x_7$ , player 1 should have chosen  $R_2$



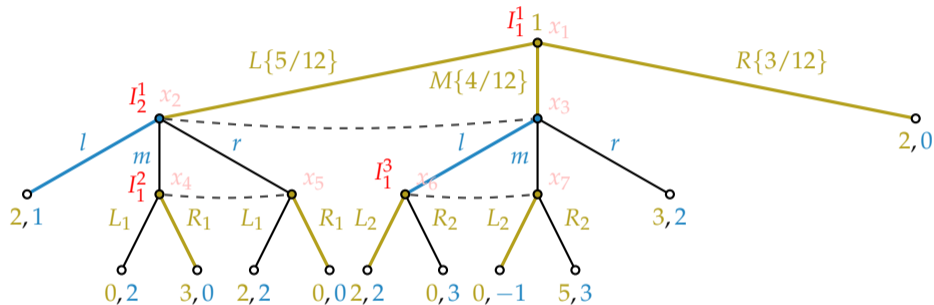
# Example: An IIEFG with perfect recall



- Player 2 at  $I_2^1$  believes the  $x_3$  is reached w.p.  $P_\sigma(x_3|I_2^1) = P_\sigma(x_3)/(P_\sigma(x_2) + P_\sigma(x_3)) = 4/9$
- Similarly  $P_\sigma(x_2|I_2^1) = 5/9$



# Example: An IIEFG with perfect recall



## Question

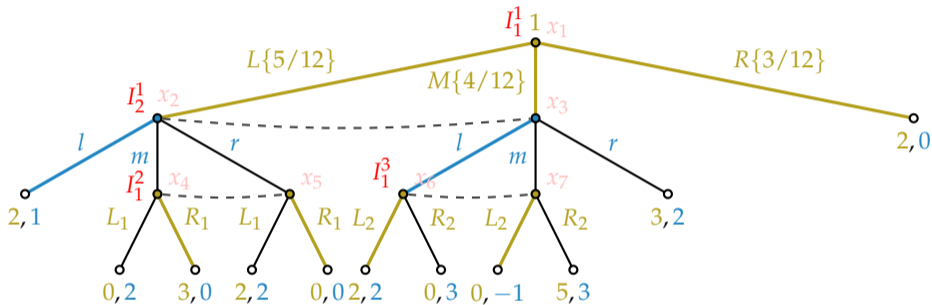
Is the action of player 2 sequentially rational w.r.t. her belief?

## Answer

By picking  $l$ , expected utility =  $5/9 \times 1 + 4/9 \times 2 = 13/9$ , larger than any other choice of action.



# Example: An IIEFG with perfect recall



## Question

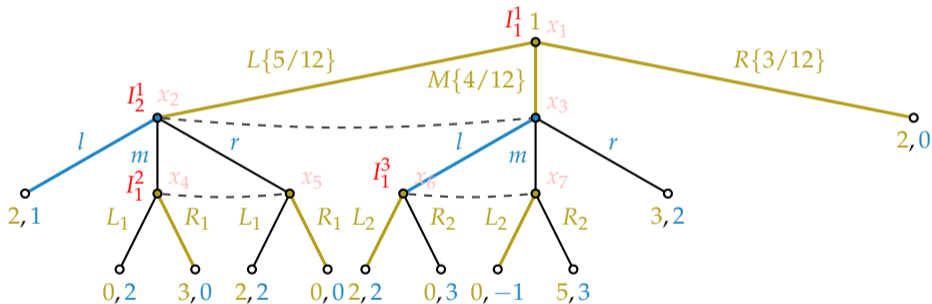
Given all information, what is the sequentially rational strategy for player 1 at  $I_1^1$

## Answer

$L, M, R$  all give the same expected utility for player 1 (utility = 2).



# Example: An IIEFG with perfect recall



Thus, mixed/behavioral strategy profile  $\sigma$  is sequentially rational for all players.



## Belief

Let the information sets of player  $i$  be  $I_i = \{I_i^1, I_i^2, I_i^3, \dots, I_i^{k(i)}\}$ .

The belief of player  $i$  is a mapping  $\mu_i^j : I_i^j \rightarrow [0, 1]$  s.t.,  $\sum_{x \in I_i^j} \mu_i^j(x) = 1$



## Belief

Let the information sets of player  $i$  be  $I_i = \{I_i^1, I_i^2, I_i^3, \dots, I_i^{k(i)}\}$ .

The belief of player  $i$  is a mapping  $\mu_i^j : I_i^j \rightarrow [0, 1]$  s.t.,  $\sum_{x \in I_i^j} \mu_i^j(x) = 1$

## Bayesian Belief

A **belief**  $\mu_i = \{\mu_i^1, \mu_i^2, \dots, \mu_i^{k(i)}\}$  of player  $i$  is **Bayesian** w.r.t. to the behavioral strategy  $\sigma$ , if it is derived from  $\sigma$  using Bayes rule, i.e.,

$$\mu_i^j(x) = P_\sigma(x) / \sum_{y \in I_i^j} P_\sigma(y), \forall x \in I_i^j, \forall j = 1, 2, 3, \dots, k(i)$$



## Sequential Rationality

A strategy  $\sigma_i$  of player  $i$  at an information set  $I_i^j$  is **sequentially rational** given  $\sigma_{-i}$  and partial belief  $\mu_i^j$  if

$$\sum_{x \in I_i^j} \mu_i^j(x) u_i(\sigma_i, \sigma_{-i} | x) \geq \sum_{x \in I_i^j} \mu_i^j(x) u_i(\sigma_i', \sigma_{-i} | x)$$



## Sequential Rationality

A strategy  $\sigma_i$  of player  $i$  at an information set  $I_i^j$  is **sequentially rational** given  $\sigma_{-i}$  and partial belief  $\mu_i^j$  if

$$\sum_{x \in I_i^j} \mu_i^j(x) u_i(\sigma_i, \sigma_{-i} | x) \geq \sum_{x \in I_i^j} \mu_i^j(x) u_i(\sigma_i', \sigma_{-i} | x)$$

- The tuple  $(\sigma, \mu)$  is sequentially rational if it is sequentially rational for every player at every information set.



## Sequential Rationality

A strategy  $\sigma_i$  of player  $i$  at an information set  $I_i^j$  is **sequentially rational** given  $\sigma_{-i}$  and partial belief  $\mu_i^j$  if

$$\sum_{x \in I_i^j} \mu_i^j(x) u_i(\sigma_i, \sigma_{-i} | x) \geq \sum_{x \in I_i^j} \mu_i^j(x) u_i(\sigma'_i, \sigma_{-i} | x)$$

- The tuple  $(\sigma, \mu)$  is sequentially rational if it is sequentially rational for every player at every information set.
- The tuple  $(\sigma, \mu)$  is also called an assessment.



## Sequential Rationality

A strategy  $\sigma_i$  of player  $i$  at an information set  $I_i^j$  is **sequentially rational** given  $\sigma_{-i}$  and partial belief  $\mu_i^j$  if

$$\sum_{x \in I_i^j} \mu_i^j(x) u_i(\sigma_i, \sigma_{-i} | x) \geq \sum_{x \in I_i^j} \mu_i^j(x) u_i(\sigma'_i, \sigma_{-i} | x)$$

- The tuple  $(\sigma, \mu)$  is sequentially rational if it is sequentially rational for every player at every information set.
- The tuple  $(\sigma, \mu)$  is also called an assessment.
- Sequential rationality is a refinement of Nash Equilibrium.



## Sequential Rationality

A strategy  $\sigma_i$  of player  $i$  at an information set  $I_i^j$  is **sequentially rational** given  $\sigma_{-i}$  and partial belief  $\mu_i^j$  if

$$\sum_{x \in I_i^j} \mu_i^j(x) u_i(\sigma_i, \sigma_{-i} | x) \geq \sum_{x \in I_i^j} \mu_i^j(x) u_i(\sigma'_i, \sigma_{-i} | x)$$

- The tuple  $(\sigma, \mu)$  is sequentially rational if it is sequentially rational for every player at every information set.
- The tuple  $(\sigma, \mu)$  is also called an assessment.
- Sequential rationality is a refinement of Nash Equilibrium.
- **The notion coincides with SPNE when applied to PIEFGs**



## Theorem

*In a PIEFG, a behavioral strategy profile  $\sigma$  is an SPNE iff the tuple  $(\sigma, \hat{\mu})$  is sequentially rational.*



## Theorem

*In a PIEFG, a behavioral strategy profile  $\sigma$  is an SPNE iff the tuple  $(\sigma, \hat{\mu})$  is sequentially rational.*

In a PIEFG, every information set is a singleton,  $\hat{\mu}$  is the degenerate distribution at that singleton.



## Theorem

*In a PIEFG, a behavioral strategy profile  $\sigma$  is an SPNE iff the tuple  $(\sigma, \hat{\mu})$  is sequentially rational.*

In a PIEFG, every information set is a singleton,  $\hat{\mu}$  is the degenerate distribution at that singleton.

## Equilibrium with Sequential Rationality

Perfect Bayesian Equilibrium: An assessment  $(\sigma, \mu)$  is PBE if  $\forall i \in N$

- $\mu_i$  is Bayesian w.r.t.  $\sigma$
- $\sigma_i$  is sequentially rational given  $\sigma_{-i}$  and  $\mu_i$



## Theorem

*In a PIEFG, a behavioral strategy profile  $\sigma$  is an SPNE iff the tuple  $(\sigma, \hat{\mu})$  is sequentially rational.*

In a PIEFG, every information set is a singleton,  $\hat{\mu}$  is the degenerate distribution at that singleton.

## Equilibrium with Sequential Rationality

Perfect Bayesian Equilibrium: An assessment  $(\sigma, \mu)$  is PBE if  $\forall i \in N$

- $\mu_i$  is Bayesian w.r.t.  $\sigma$
- $\sigma_i$  is sequentially rational given  $\sigma_{-i}$  and  $\mu_i$
  
- Often represented only with  $\sigma$ , since  $\mu$  is obtained from  $\sigma$



## Theorem

*In a PIEFG, a behavioral strategy profile  $\sigma$  is an SPNE iff the tuple  $(\sigma, \hat{\mu})$  is sequentially rational.*

In a PIEFG, every information set is a singleton,  $\hat{\mu}$  is the degenerate distribution at that singleton.

## Equilibrium with Sequential Rationality

Perfect Bayesian Equilibrium: An assessment  $(\sigma, \mu)$  is PBE if  $\forall i \in N$

- $\mu_i$  is Bayesian w.r.t.  $\sigma$
  - $\sigma_i$  is sequentially rational given  $\sigma_{-i}$  and  $\mu_i$
- 
- Often represented only with  $\sigma$ , since  $\mu$  is obtained from  $\sigma$
  - Self-enforcing (like the SPNE) in a Bayesian way.

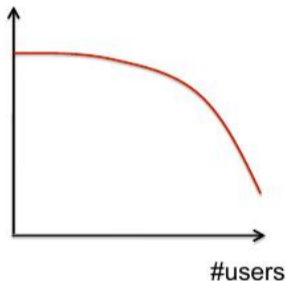


- ▶ Equilibrium in IIEFGs
- ▶ Game Theory in Practice: P2P File Sharing
- ▶ Bayesian Games
- ▶ Strategy, Utility in Bayesian Games
- ▶ Equilibrium in Bayesian Games
- ▶ Examples in Bayesian Equilibrium

# Peer to Peer<sup>1</sup>

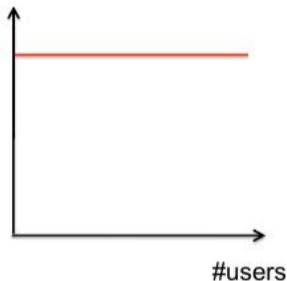


download rate



traditional

download rate



P2P

<sup>1</sup>Slides of this section are adapted from CS186, Harvard

# Desired Properties and Terminology



- Scalability

**Terminology:**

# Desired Properties and Terminology



- Scalability
- Failure resilience

**Terminology:**

# Desired Properties and Terminology



- Scalability
- Failure resilience

## Terminology:

- **Protocol:** messages that can be sent, actions that can be taken over the network

# Desired Properties and Terminology



- Scalability
- Failure resilience

## Terminology:

- **Protocol:** messages that can be sent, actions that can be taken over the network
- **Client:** a particular process for sending messages, taking actions

# Desired Properties and Terminology



- Scalability
- Failure resilience

## Terminology:

- **Protocol:** messages that can be sent, actions that can be taken over the network
- **Client:** a particular process for sending messages, taking actions
- **Reference client:** particular implementation

# Desired Properties and Terminology



- Scalability
- Failure resilience

## Terminology:

- **Protocol:** messages that can be sent, actions that can be taken over the network
- **Client:** a particular process for sending messages, taking actions
- **Reference client:** particular implementation
- **Peer**



## **Napster** (1999 - 2001)

- Centralized database
- Users download music from each other



## **Napster** (1999 - 2001)

- Centralized database
- Users download music from each other

## **Gnutella** (2000 - )

- Get list of IP addresses of peers from set of known peers (no server)
- To get a file: Query message broadcast by peer A to known peers
- Query response: sent by B if B has the desired file (routed back to requestor)
- A can then download directly from B

# The File Sharing Game



		Player 2	
		Share	Free-ride
Player 1	Share	2,2	-1,3
	Free-ride	3,-1	0,0

(Gnutella) File Sharing Game

# The File Sharing Game (Contd.)

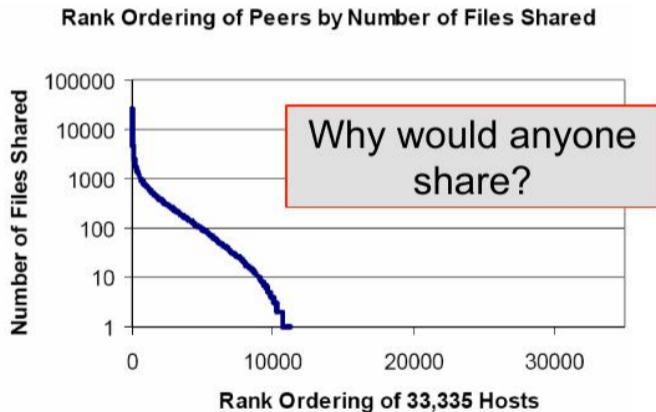


Image courtesy: Adar and Huberman (2000)

# Incentives for Client Developers



- Client developers can ensure file sharing
- But competition among the developers

# Incentives for Client Developers



- Client developers can ensure file sharing
- But competition among the developers
- 85% peers free-riding by 2005; Gnutella less than 1% of ww P2P traffic by 2013
- Few other P2P systems met the same fate



## BitTorrent (2001 - )

- Approx 85% of P2P traffic in US
- File sharing
- Also used for S/W distribution (e.g., Linux)



## BitTorrent (2001 - )

- Approx 85% of P2P traffic in US
- File sharing
- Also used for S/W distribution (e.g., Linux)

## Key innovations

- Break file into pieces: A repeated game!
- “If you let me download, I’ll reciprocate.”

# BitTorrent Schematic

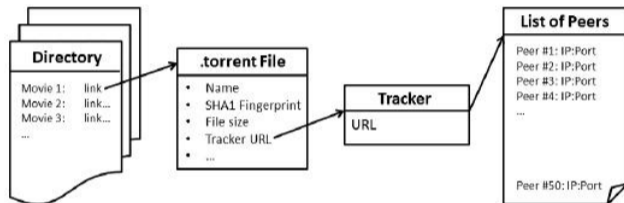


Figure 5.4.: Starting a download process in the BitTorrent protocol: 1) A user goes to a searchable directory to find a link to a .torrent file corresponding to the desired content; 2) the .torrent file contains metadata about the content, in particular the URL of a tracker; 3) the tracker provides a list of peers participating in the swarm for the content (i.e., their IP address and port); 4) the user's BitTorrent client can now contact all these peers and download content.

# BitTorrent Optimistic Unchoking Algorithm



Tracker is a centralized entity that controls the traffic, tracks the connection between peers and their speed of upload, download etc.

# BitTorrent Optimistic Unchoking Algorithm



Tracker is a centralized entity that controls the traffic, tracks the connection between peers and their speed of upload, download etc.

## Reference Client Protocol:

- Set a threshold  $r$  of uploading speed (typically the third maximum speed in the recent past)

# BitTorrent Optimistic Unchoking Algorithm



Tracker is a centralized entity that controls the traffic, tracks the connection between peers and their speed of upload, download etc.

## Reference Client Protocol:

- Set a threshold  $r$  of uploading speed (typically the third maximum speed in the recent past)
- If a peer  $j$  uploaded to  $i$  at a rate  $\geq r$ , unchoke  $j$  in the next period

# BitTorrent Optimistic Unchoking Algorithm



Tracker is a centralized entity that controls the traffic, tracks the connection between peers and their speed of upload, download etc.

## Reference Client Protocol:

- Set a threshold  $r$  of uploading speed (typically the third maximum speed in the recent past)
- If a peer  $j$  uploaded to  $i$  at a rate  $\geq r$ , unchoke  $j$  in the next period
- If a peer  $j$  uploaded to  $i$  at a rate  $< r$ , choke  $j$  in the next period

# BitTorrent Optimistic Unchoking Algorithm



Tracker is a centralized entity that controls the traffic, tracks the connection between peers and their speed of upload, download etc.

## Reference Client Protocol:

- Set a threshold  $r$  of uploading speed (typically the third maximum speed in the recent past)
- If a peer  $j$  uploaded to  $i$  at a rate  $\geq r$ , unchoke  $j$  in the next period
- If a peer  $j$  uploaded to  $i$  at a rate  $< r$ , choke  $j$  in the next period
- Every three time periods, optimistically unchoke a random peer from the neighborhood who is currently choked, and leave that peer unchoked for three time periods.

# BitTorrent Optimistic Unchoking Algorithm



Tracker is a centralized entity that controls the traffic, tracks the connection between peers and their speed of upload, download etc.

## Reference Client Protocol:

- Set a threshold  $r$  of uploading speed (typically the third maximum speed in the recent past)
- If a peer  $j$  uploaded to  $i$  at a rate  $\geq r$ , unchoke  $j$  in the next period
- If a peer  $j$  uploaded to  $i$  at a rate  $< r$ , choke  $j$  in the next period
- Every three time periods, optimistically unchoke a random peer from the neighborhood who is currently choked, and leave that peer unchoked for three time periods.

Forcing a repeated game by fragmenting the files

# BitTorrent Optimistic Unchoking Algorithm



Tracker is a centralized entity that controls the traffic, tracks the connection between peers and their speed of upload, download etc.

## Reference Client Protocol:

- Set a threshold  $r$  of uploading speed (typically the third maximum speed in the recent past)
- If a peer  $j$  uploaded to  $i$  at a rate  $\geq r$ , unchoke  $j$  in the next period
- If a peer  $j$  uploaded to  $i$  at a rate  $< r$ , choke  $j$  in the next period
- Every three time periods, optimistically unchoke a random peer from the neighborhood who is currently choked, and leave that peer unchoked for three time periods.

Forcing a repeated game by fragmenting the files

The leecher-seeder game is a repeated Prisoners' Dilemma

# BitTorrent Optimistic Unchoking Algorithm



Tracker is a centralized entity that controls the traffic, tracks the connection between peers and their speed of upload, download etc.

## Reference Client Protocol:

- Set a threshold  $r$  of uploading speed (typically the third maximum speed in the recent past)
- If a peer  $j$  uploaded to  $i$  at a rate  $\geq r$ , unchoke  $j$  in the next period
- If a peer  $j$  uploaded to  $i$  at a rate  $< r$ , choke  $j$  in the next period
- Every three time periods, optimistically unchoke a random peer from the neighborhood who is currently choked, and leave that peer unchoked for three time periods.

Forcing a repeated game by fragmenting the files

The leecher-seeder game is a repeated Prisoners' Dilemma

Strategy of the seeder is tit-for-tat



Illustration



- How often to contact tracker?
- Which pieces to reveal?
- How many upload slots, which peers to unchoke, at what speed?
- What data to allow others to download?
- Possible goals: min upload, max download speed, some balance

# Attacks on BitTorrent



- BitThief
- Strategic piece revealer
- BitTyrant



- Goal: download files without uploading
- Keep asking for peers from tracker, grow neighborhood quickly
- Exploit the optimistic unchoking part
- Never upload!



- Goal: download files without uploading
- Keep asking for peers from tracker, grow neighborhood quickly
- Exploit the optimistic unchoking part
- Never upload!
- **Fix: modify the tracker (block same IP address within 30 minutes).**

Ref: Locher et al., "Free Riding in BitTorrent is Cheap", HotNets 2006

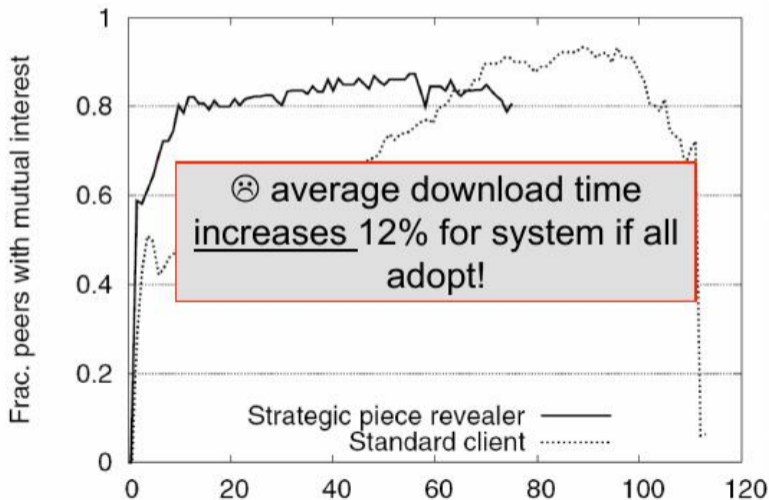
# Strategic Piece Revealer



- Reference client: tell neighbors about new pieces, use “rarest-first” to request
- Manipulator strategy: reveal most common piece that reciprocating peer does not have!
- Try to protect a monopoly, keep others interested

Ref: Levin et al., “BitTorrent is an Auction: Analyzing and Improving BitTorrent’s Incentives”, SIGCOMM 2008

# Strategic Piece Revealer





- P2P demonstrates importance of game-theory in computer systems



- P2P demonstrates importance of game-theory in computer systems
- Early systems were easily manipulated



- P2P demonstrates importance of game-theory in computer systems
- Early systems were easily manipulated
- BitTorrent's innovation was to break files into pieces, enabling TitForTat.



- P2P demonstrates importance of game-theory in computer systems
- Early systems were easily manipulated
- BitTorrent's innovation was to break files into pieces, enabling TitForTat.
- Still some vulnerabilities, but generally very successful example of incentive-based protocol design.



- ▶ Equilibrium in IIEFGs
- ▶ Game Theory in Practice: P2P File Sharing
- ▶ **Bayesian Games**
- ▶ Strategy, Utility in Bayesian Games
- ▶ Equilibrium in Bayesian Games
- ▶ Examples in Bayesian Equilibrium



## Games

- **Non-cooperative games**



## Games

- **Non-cooperative games**
  - **Complete information** - Players **deterministically** know which game they are playing



## Games

- **Non-cooperative games**
  - **Complete information** - Players **deterministically** know which game they are playing
    - Normal form games
      - Appropriate for simultaneous move single-stage games
      - Equilibrium notions: SDSE, WDSE, PSNE, MSNE, Correlated



## Games

- **Non-cooperative games**

- **Complete information** - Players **deterministically** know which game they are playing

- Normal form games

- Appropriate for simultaneous move single-stage games

- Equilibrium notions: SDSE, WDSE, PSNE, MSNE, Correlated

- Extensive form games

- Appropriate for multi-stage games

- Equilibrium notions: SPNE (PIEFG), mixed and behavioral strategies (IIEFG), PBE



## Games

- **Non-cooperative games**

- **Complete information** - Players **deterministically** know which game they are playing
  - Normal form games
    - Appropriate for simultaneous move single-stage games
    - Equilibrium notions: SDSE, WDSE, PSNE, MSNE, Correlated
  - Extensive form games
    - Appropriate for multi-stage games
    - Equilibrium notions: SPNE (PIEFG), mixed and behavioral strategies (IIEFG), PBE
- **Incomplete information** - Players **do not deterministically** know which game they are playing



## Games

- **Non-cooperative games**
  - **Complete information** - Players **deterministically** know which game they are playing
    - Normal form games  
Appropriate for simultaneous move single-stage games  
Equilibrium notions: SDSE, WDSE, PSNE, MSNE, Correlated
    - Extensive form games  
Appropriate for multi-stage games  
Equilibrium notions: SPNE (PIEFG), mixed and behavioral strategies (IIEFG), PBE
  - **Incomplete information** - Players **do not deterministically** know which game they are playing
- **Cooperative games** - Players form coalitions and utilities are defined over coalitions



## Games

- **Non-cooperative games**
  - **Complete information** - Players **deterministically** know which game they are playing
    - Normal form games  
Appropriate for simultaneous move single-stage games  
Equilibrium notions: SDSE, WDSE, PSNE, MSNE, Correlated
    - Extensive form games  
Appropriate for multi-stage games  
Equilibrium notions: SPNE (PIEFG), mixed and behavioral strategies (IIEFG), PBE
  - **Incomplete information** - Players **do not deterministically** know which game they are playing
- **Cooperative games** - Players form coalitions and utilities are defined over coalitions
- Other types of games - repeated, stochastic etc.

# Games with Incomplete Information



## Games with Complete Information

- Players deterministically know the game they are playing
- There can be some chance moves but probabilities are known

## Games with Incomplete information

# Games with Incomplete Information



## Games with Complete Information

- Players deterministically know the game they are playing
- There can be some chance moves but probabilities are known

## Games with Incomplete information

- Players do not know deterministically know which game they are playing

# Games with Incomplete Information



## Games with Complete Information

- Players deterministically know the game they are playing
- There can be some chance moves but probabilities are known

## Games with Incomplete information

- Players do not know deterministically know which game they are playing
- They receive **private signals / types**

# Games with Incomplete Information



## Games with Complete Information

- Players deterministically know the game they are playing
- There can be some chance moves but probabilities are known

## Games with Incomplete information

- Players do not know deterministically know which game they are playing
- They receive **private signals / types**
- To discuss: a special subclass called games with incomplete information with **common priors** (Harsanyi 1967)

# Games with Incomplete Information



## Games with Complete Information

- Players deterministically know the game they are playing
- There can be some chance moves but probabilities are known

## Games with Incomplete information

- Players do not know deterministically know which game they are playing
- They receive **private signals / types**
- To discuss: a special subclass called games with incomplete information with **common priors** (Harsanyi 1967)
- Also called **Bayesian games**

# Bayesian Games: Example



Football game (two competing teams)

# Bayesian Games: Example



Football game (two competing teams)

- Each can choose a gameplan: aim to win (Aggressive) or aim to draw (Passive)

# Bayesian Games: Example



Football game (two competing teams)

- Each can choose a gameplan: aim to win (Aggressive) or aim to draw (Passive)
- We will call the gameplan as their **type** which are private signals to them, often caused by external factors, e.g., weather conditions, player injuries, ground conditions etc.

# Bayesian Games: Example



Football game (two competing teams)

- Each can choose a gameplan: aim to win (Aggressive) or aim to draw (Passive)
- We will call the gameplan as their **type** which are private signals to them, often caused by external factors, e.g., weather conditions, player injuries, ground conditions etc.
- There are four possible type profiles in this example AA, AP, PA, PP.

# Bayesian Games: Example



Football game (two competing teams)

- Each can choose a gameplan: aim to win (Aggressive) or aim to draw (Passive)
- We will call the gameplan as their **type** which are private signals to them, often caused by external factors, e.g., weather conditions, player injuries, ground conditions etc.
- There are four possible type profiles in this example AA, AP, PA, PP.
- The payoff matrices differ as follows (payoff for PA is symmetrically opposite to AP).

# Bayesian Games: Example



Football game (two competing teams)

- Each can choose a gameplan: aim to win (Aggressive) or aim to draw (Passive)
- We will call the gameplan as their **type** which are private signals to them, often caused by external factors, e.g., weather conditions, player injuries, ground conditions etc.
- There are four possible type profiles in this example AA, AP, PA, PP.
- The payoff matrices differ as follows (payoff for PA is symmetrically opposite to AP).

		FRA	
		ATT	DEF
ARG	ATT	1,1	2,0
	DEF	0,2	0,0

AA profile

# Bayesian Games: Example



Football game (two competing teams)

- Each can choose a gameplan: aim to win (Aggressive) or aim to draw (Passive)
- We will call the gameplan as their **type** which are private signals to them, often caused by external factors, e.g., weather conditions, player injuries, ground conditions etc.
- There are four possible type profiles in this example AA, AP, PA, PP.
- The payoff matrices differ as follows (payoff for PA is symmetrically opposite to AP).

		FRA	
		ATT	DEF
ARG	ATT	1,1	2,0
	DEF	0,2	0,0

AA profile

		FRA	
		ATT	DEF
ARG	ATT	2,0	2,1
	DEF	0,1	0,0

AP profile



# Bayesian Games: Example

Football game (two competing teams)

- Each can choose a gameplan: aim to win (Aggressive) or aim to draw (Passive)
- We will call the gameplan as their **type** which are private signals to them, often caused by external factors, e.g., weather conditions, player injuries, ground conditions etc.
- There are four possible type profiles in this example AA, AP, PA, PP.
- The payoff matrices differ as follows (payoff for PA is symmetrically opposite to AP).

		FRA	
		ATT	DEF
ARG	ATT	1,1	2,0
	DEF	0,2	0,0

AA profile

		FRA	
		ATT	DEF
ARG	ATT	2,0	2,1
	DEF	0,1	0,0

AP profile

		FRA	
		ATT	DEF
ARG	ATT	0,0	1,0
	DEF	0,1	-1,-1

PP profile



## Assumptions

- The probabilities of choosing different games (or type profiles) come from a **common prior** distribution.



## Assumptions

- The probabilities of choosing different games (or type profiles) come from a **common prior** distribution.
- The common prior is common knowledge



## Assumptions

- The probabilities of choosing different games (or type profiles) come from a **common prior** distribution.
- The common prior is common knowledge

## Definition

A Bayesian game is represented by  $\langle N, (\Theta_i)_{i \in N}, P, (\Gamma_\theta)_{\theta \in (\times_{i \in N} \Theta_i)} \rangle$

- $N$ : set of players



## Assumptions

- The probabilities of choosing different games (or type profiles) come from a **common prior** distribution.
- The common prior is common knowledge

## Definition

A Bayesian game is represented by  $\langle N, (\Theta_i)_{i \in N}, P, (\Gamma_\theta)_{\theta \in (\times_{i \in N} \Theta_i)} \rangle$

- $N$ : set of players
- $\Theta_i$ : set of types of player  $i$



## Assumptions

- The probabilities of choosing different games (or type profiles) come from a **common prior** distribution.
- The common prior is common knowledge

## Definition

A Bayesian game is represented by  $\langle N, (\Theta_i)_{i \in N}, P, (\Gamma_\theta)_{\theta \in (\times_{i \in N} \Theta_i)} \rangle$

- $N$ : set of players
- $\Theta_i$ : set of types of player  $i$
- $P$ : common prior distribution over  $\Theta = \times_{i \in N} \Theta_i$   
s.t.  $\sum_{\theta_{-i} \in \Theta_{-i}} P(\theta_i, \theta_{-i}) > 0, \quad \forall \theta_i \in \Theta_i, \forall i \in N$   
i.e., marginals for every type is positive (otherwise we can prune the type set)



## Assumptions

- The probabilities of choosing different games (or type profiles) come from a **common prior** distribution.
- The common prior is common knowledge

## Definition

A Bayesian game is represented by  $\langle N, (\Theta_i)_{i \in N}, P, (\Gamma_\theta)_{\theta \in (\times_{i \in N} \Theta_i)} \rangle$

- $N$ : set of players
- $\Theta_i$ : set of types of player  $i$
- $P$ : common prior distribution over  $\Theta = \times_{i \in N} \Theta_i$   
s.t.  $\sum_{\theta_{-i} \in \Theta_{-i}} P(\theta_i, \theta_{-i}) > 0, \quad \forall \theta_i \in \Theta_i, \forall i \in N$   
i.e., marginals for every type is positive (otherwise we can prune the type set)
- $\Gamma_\theta$ : NFG for the type profile  $\theta \in \Theta$  i.e.,  $\Gamma_\theta = \langle N, (A_i(\theta))_{i \in N}, (u_i(\theta))_{i \in N} \rangle$   
 $u_i : A \times \Theta \rightarrow \mathbb{R}, A = \times_{i \in N} A_i$  [We assume  $A_i(\theta) = A_i, \forall \theta$ ]



## Stages of a Bayesian game

- $\theta = (\theta_i, \theta_{-i})$  is chosen randomly according to the common prior  $P$



## Stages of a Bayesian game

- $\theta = (\theta_i, \theta_{-i})$  is chosen randomly according to the common prior  $P$
- Each player observes her own type  $\theta_i$



## Stages of a Bayesian game

- $\theta = (\theta_i, \theta_{-i})$  is chosen randomly according to the common prior  $P$
- Each player observes her own type  $\theta_i$
- Player  $i$  picks action  $a_i \in A_i, \forall i \in N$



## Stages of a Bayesian game

- $\theta = (\theta_i, \theta_{-i})$  is chosen randomly according to the common prior  $P$
- Each player observes her own type  $\theta_i$
- Player  $i$  picks action  $a_i \in A_i, \forall i \in N$
- Player  $i$  realizes a payoff of  $u_i(a_i, a_{-i}; \theta_i, \theta_{-i})$



- ▶ Equilibrium in IIEFGs
- ▶ Game Theory in Practice: P2P File Sharing
- ▶ Bayesian Games
- ▶ **Strategy, Utility in Bayesian Games**
- ▶ Equilibrium in Bayesian Games
- ▶ Examples in Bayesian Equilibrium



## Definition

Strategy is a plan to map type to action.

$$s_i : \Theta_i \rightarrow A_i$$

$$\sigma_i : \Theta_i \rightarrow \Delta A_i$$

**Pure**

**Mixed**



## Definition

Strategy is a plan to map type to action.

$$s_i : \Theta_i \rightarrow A_i$$

$$\sigma_i : \Theta_i \rightarrow \Delta A_i$$

**Pure**

**Mixed**

The player can experience its utility in two stages for Bayesian games (depending on the realization of  $\theta_i$ ).



## Definition

Strategy is a plan to map type to action.

$$s_i : \Theta_i \rightarrow A_i$$

$$\sigma_i : \Theta_i \rightarrow \Delta A_i$$

**Pure**

**Mixed**

The player can experience its utility in two stages for Bayesian games (depending on the realization of  $\theta_i$ ).

- Ex-ante utility



## Definition

Strategy is a plan to map type to action.

$$s_i : \Theta_i \rightarrow A_i$$

$$\sigma_i : \Theta_i \rightarrow \Delta A_i$$

**Pure**

**Mixed**

The player can experience its utility in two stages for Bayesian games (depending on the realization of  $\theta_i$ ).

- Ex-ante utility
- Ex-interim utility



## Definition

Strategy is a plan to map type to action.

$$s_i : \Theta_i \rightarrow A_i$$

$$\sigma_i : \Theta_i \rightarrow \Delta A_i$$

**Pure**

**Mixed**

The player can experience its utility in two stages for Bayesian games (depending on the realization of  $\theta_i$ ).

- Ex-ante utility
- Ex-interim utility
- Ex-post utility (complete information game)



## Definition (Ex-ante utility)

Expected utility **before** observing own type.

$$\begin{aligned} u_i(\sigma) &= \sum_{\theta \in \Theta} P(\theta) u_i(\sigma(\theta); \theta) \\ &= \sum_{\theta \in \Theta} P(\theta) \sum_{(a_1, a_2, \dots, a_n) \in A} \prod_{j \in N} \sigma_j(\theta_j)[a_j] u_i(a_1, \dots, a_n; \theta_1, \dots, \theta_n) \end{aligned}$$



## Definition (Ex-ante utility)

Expected utility **before** observing own type.

$$\begin{aligned} u_i(\sigma) &= \sum_{\theta \in \Theta} P(\theta) u_i(\sigma(\theta); \theta) \\ &= \sum_{\theta \in \Theta} P(\theta) \sum_{(a_1, a_2, \dots, a_n) \in A} \prod_{j \in N} \sigma_j(\theta_j)[a_j] u_i(a_1, \dots, a_n; \theta_1, \dots, \theta_n) \end{aligned}$$

The **belief** of player  $i$  over others' types changes after observing her own type  $\theta_i$ :

$$P(\theta_{-i} | \theta_i) = \frac{P(\theta_i, \theta_{-i})}{\sum_{\tilde{\theta}_{-i} \in \Theta_{-i}} P(\theta_i, \tilde{\theta}_{-i})} \quad \text{Bayes rule}$$



## Definition (Ex-ante utility)

Expected utility **before** observing own type.

$$\begin{aligned} u_i(\sigma) &= \sum_{\theta \in \Theta} P(\theta) u_i(\sigma(\theta); \theta) \\ &= \sum_{\theta \in \Theta} P(\theta) \sum_{(a_1, a_2, \dots, a_n) \in A} \prod_{j \in N} \sigma_j(\theta_j) [a_j] u_i(a_1, \dots, a_n; \theta_1, \dots, \theta_n) \end{aligned}$$

The **belief** of player  $i$  over others' types changes after observing her own type  $\theta_i$ :

$$P(\theta_{-i} | \theta_i) = \frac{P(\theta_i, \theta_{-i})}{\sum_{\tilde{\theta}_{-i} \in \Theta_{-i}} P(\theta_i, \tilde{\theta}_{-i})} \quad \text{Bayes rule}$$

This is why we needed every marginal to be positive



## Definition (Ex-interim utility)

Expected utility after observing one's own type.

$$u_i(\sigma|\theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} P(\theta_{-i}|\theta_i) u_i(\sigma(\theta); \theta)$$



## Definition (Ex-interim utility)

Expected utility after observing one's own type.

$$u_i(\sigma|\theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} P(\theta_{-i}|\theta_i) u_i(\sigma(\theta); \theta)$$

**Special Case:** for independent types, observing  $\theta_i$  does not give any information on  $\theta_{-i}$



## Definition (Ex-interim utility)

Expected utility after observing one's own type.

$$u_i(\sigma|\theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} P(\theta_{-i}|\theta_i) u_i(\sigma(\theta); \theta)$$

**Special Case:** for independent types, observing  $\theta_i$  does not give any information on  $\theta_{-i}$

Relation between the two utilities is given by

$$u_i(\sigma) = \sum_{\theta_i \in \Theta_i} P(\theta_i) u_i(\sigma|\theta_i)$$

# Example 1: Two Player Bargaining Game



- Player 1 : seller, type : price at which he is willing to sell

# Example 1: Two Player Bargaining Game



- Player 1 : seller, type : price at which he is willing to sell
- Player 2 : buyer, type : price at which he is willing to buy



## Example 1: Two Player Bargaining Game

- Player 1 : seller, type : price at which he is willing to sell
- Player 2 : buyer, type : price at which he is willing to buy
- $\Theta_1 = \Theta_2 = \{1, 2, \dots, 100\}$ ,  $A_1 = A_2 = \{1, 2, \dots, 100\}$

# Example 1: Two Player Bargaining Game



- Player 1 : seller, type : price at which he is willing to sell
- Player 2 : buyer, type : price at which he is willing to buy
- $\Theta_1 = \Theta_2 = \{1, 2, \dots, 100\}$ ,  $A_1 = A_2 = \{1, 2, \dots, 100\}$
- If the bid of the seller is smaller or equal to that of the buyer, trade happens at a price average of the two bids. Else, trade does not happen.

## Example 1: Two Player Bargaining Game (Contd.)



Suppose type generation is independent and uniform over  $\Theta_1, \Theta_2$  respectively,

$$P(\theta_2|\theta_1) = P(\theta_2) = \frac{1}{100}, \forall \theta_1, \theta_2$$

$$P(\theta_1|\theta_2) = P(\theta_1) = \frac{1}{100}, \forall \theta_1, \theta_2$$



## Example 1: Two Player Bargaining Game (Contd.)

Suppose type generation is independent and uniform over  $\Theta_1, \Theta_2$  respectively,

$$P(\theta_2|\theta_1) = P(\theta_2) = \frac{1}{100}, \forall \theta_1, \theta_2$$

$$P(\theta_1|\theta_2) = P(\theta_1) = \frac{1}{100}, \forall \theta_1, \theta_2$$

$$u_1(a_1, a_2; \theta_1, \theta_2) = \begin{cases} \frac{a_1+a_2}{2} - \theta_1 & \text{if } a_2 \geq a_1 \\ 0 & \text{otherwise} \end{cases}$$

$$u_2(a_1, a_2; \theta_1, \theta_2) = \begin{cases} \theta_2 - \frac{a_1+a_2}{2} & \text{if } a_2 \geq a_1 \\ 0 & \text{otherwise} \end{cases}$$



## Example 1: Two Player Bargaining Game (Contd.)

Suppose type generation is independent and uniform over  $\Theta_1, \Theta_2$  respectively,

$$P(\theta_2|\theta_1) = P(\theta_2) = \frac{1}{100}, \forall \theta_1, \theta_2$$

$$P(\theta_1|\theta_2) = P(\theta_1) = \frac{1}{100}, \forall \theta_1, \theta_2$$

$$u_1(a_1, a_2; \theta_1, \theta_2) = \begin{cases} \frac{a_1+a_2}{2} - \theta_1 & \text{if } a_2 \geq a_1 \\ 0 & \text{otherwise} \end{cases}$$

$$u_2(a_1, a_2; \theta_1, \theta_2) = \begin{cases} \theta_2 - \frac{a_1+a_2}{2} & \text{if } a_2 \geq a_1 \\ 0 & \text{otherwise} \end{cases}$$

Common Prior :  $P(\theta_1, \theta_2) = \frac{1}{10000}, \forall \theta_1, \theta_2$

## Example 2: Sealed Bid Auction



Two players, both willing to buy an object. Their values  $\theta_i$  and bids  $b_i$  lie in  $[0, 1]$ .

## Example 2: Sealed Bid Auction



Two players, both willing to buy an object. Their values  $\theta_i$  and bids  $b_i$  lie in  $[0, 1]$ .

**Allocation Function:**

$$O_1(b_1, b_2) = \begin{cases} 1 & \text{if } b_1 \geq b_2 \\ 0 & \text{ow} \end{cases} \quad O_2(b_1, b_2) = \begin{cases} 1 & \text{if } b_2 > b_1 \\ 0 & \text{ow} \end{cases}$$

## Example 2: Sealed Bid Auction



Two players, both willing to buy an object. Their values  $\theta_i$  and bids  $b_i$  lie in  $[0, 1]$ .

**Allocation Function:**

$$O_1(b_1, b_2) = \begin{cases} 1 & \text{if } b_1 \geq b_2 \\ 0 & \text{ow} \end{cases} \quad O_2(b_1, b_2) = \begin{cases} 1 & \text{if } b_2 > b_1 \\ 0 & \text{ow} \end{cases}$$

**Beliefs:**

$$f(\theta_2|\theta_1) = 1, \forall \theta_1, \theta_2$$

$$f(\theta_1|\theta_2) = 1, \forall \theta_1, \theta_2$$

$$f(\theta_1, \theta_2) = 1, \forall \theta_1, \theta_2$$



## Example 2: Sealed Bid Auction

Two players, both willing to buy an object. Their values  $\theta_i$  and bids  $b_i$  lie in  $[0, 1]$ .

**Allocation Function:**

$$O_1(b_1, b_2) = \begin{cases} 1 & \text{if } b_1 \geq b_2 \\ 0 & \text{ow} \end{cases} \quad O_2(b_1, b_2) = \begin{cases} 1 & \text{if } b_2 > b_1 \\ 0 & \text{ow} \end{cases}$$

**Beliefs:**

$$f(\theta_2|\theta_1) = 1, \forall \theta_1, \theta_2$$

$$f(\theta_1|\theta_2) = 1, \forall \theta_1, \theta_2$$

$$f(\theta_1, \theta_2) = 1, \forall \theta_1, \theta_2$$

$$u_i(b_1, b_2; \theta_1, \theta_2) = O_i(b_1, b_2)(\theta_i - b_i)$$

**Winner pays his/her bid.**



- ▶ Equilibrium in IIEFGs
- ▶ Game Theory in Practice: P2P File Sharing
- ▶ Bayesian Games
- ▶ Strategy, Utility in Bayesian Games
- ▶ Equilibrium in Bayesian Games**
- ▶ Examples in Bayesian Equilibrium

# Equilibrium concepts in Bayesian games



Ex-ante: before observing her own type

**Nash Equilibrium**  $(\sigma^*, P)$ :  $u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i', \sigma_{-i}^*), \forall \sigma_i', \forall i \in N$

# Equilibrium concepts in Bayesian games



Ex-ante: before observing her own type

**Nash Equilibrium**  $(\sigma^*, P)$ :  $u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i', \sigma_{-i}^*), \forall \sigma_i', \forall i \in N$

Ex-interim: after observing her own type

**Bayesian Equilibrium**  $(\sigma^*, P)$ :  $u_i(\sigma_i^*(\theta_i), \sigma_{-i}^*|\theta_i) \geq u_i(\sigma_i'(\theta_i), \sigma_{-i}^*|\theta_i), \forall \sigma_i', \forall \theta_i \in \Theta_i, \forall i \in N$

$$u_i(\sigma) = \sum_{\theta \in \Theta} P(\theta) u_i(\sigma(\theta); \theta), \quad u_i(\sigma|\theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} P(\theta_{-i}|\theta_i) u_i(\sigma(\theta); \theta)$$

# Equilibrium concepts in Bayesian games



Ex-ante: before observing her own type

**Nash Equilibrium**  $(\sigma^*, P)$ :  $u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i', \sigma_{-i}^*), \forall \sigma_i', \forall i \in N$

Ex-interim: after observing her own type

**Bayesian Equilibrium**  $(\sigma^*, P)$ :  $u_i(\sigma_i^*(\theta_i), \sigma_{-i}^*|\theta_i) \geq u_i(\sigma_i'(\theta_i), \sigma_{-i}^*|\theta_i), \forall \sigma_i', \forall \theta_i \in \Theta_i, \forall i \in N$

$$u_i(\sigma) = \sum_{\theta \in \Theta} P(\theta) u_i(\sigma(\theta); \theta), \quad u_i(\sigma|\theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} P(\theta_{-i}|\theta_i) u_i(\sigma(\theta); \theta)$$

- The RHS of the definitions can be replaced by a pure strategy  $a_i, \forall a_i \in A_i$ . The reason is exactly the same as that of MSNE (these definitions are equivalent)

# Equilibrium concepts in Bayesian games



Ex-ante: before observing her own type

**Nash Equilibrium**  $(\sigma^*, P)$ :  $u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i', \sigma_{-i}^*), \forall \sigma_i', \forall i \in N$

Ex-interim: after observing her own type

**Bayesian Equilibrium**  $(\sigma^*, P)$ :  $u_i(\sigma_i^*(\theta_i), \sigma_{-i}^*|\theta_i) \geq u_i(\sigma_i'(\theta_i), \sigma_{-i}^*|\theta_i), \forall \sigma_i', \forall \theta_i \in \Theta_i, \forall i \in N$

$$u_i(\sigma) = \sum_{\theta \in \Theta} P(\theta) u_i(\sigma(\theta); \theta), \quad u_i(\sigma|\theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} P(\theta_{-i}|\theta_i) u_i(\sigma(\theta); \theta)$$

- The RHS of the definitions can be replaced by a pure strategy  $a_i, \forall a_i \in A_i$ . The reason is exactly the same as that of MSNE (these definitions are equivalent)
- NE takes expectation over  $P(\theta)$

# Equilibrium concepts in Bayesian games



Ex-ante: before observing her own type

**Nash Equilibrium**  $(\sigma^*, P)$ :  $u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i', \sigma_{-i}^*), \forall \sigma_i', \forall i \in N$

Ex-interim: after observing her own type

**Bayesian Equilibrium**  $(\sigma^*, P)$ :  $u_i(\sigma_i^*(\theta_i), \sigma_{-i}^*|\theta_i) \geq u_i(\sigma_i'(\theta_i), \sigma_{-i}^*|\theta_i), \forall \sigma_i', \forall \theta_i \in \Theta_i, \forall i \in N$

$$u_i(\sigma) = \sum_{\theta \in \Theta} P(\theta) u_i(\sigma(\theta); \theta), \quad u_i(\sigma|\theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} P(\theta_{-i}|\theta_i) u_i(\sigma(\theta); \theta)$$

- The RHS of the definitions can be replaced by a pure strategy  $a_i, \forall a_i \in A_i$ . The reason is exactly the same as that of MSNE (these definitions are equivalent)
- NE takes expectation over  $P(\theta)$
- BE takes expectation over  $P(\theta_{-i}|\theta_i)$

# Equivalence of equilibrium concepts



## Theorem

*In finite Bayesian games, a strategy profile is Bayesian Equilibrium iff it is a Nash equilibrium*

# Equivalence of equilibrium concepts



## Theorem

*In finite Bayesian games, a strategy profile is Bayesian Equilibrium iff it is a Nash equilibrium*

## Proof.

For the forward direction, suppose  $(\sigma^*, P)$  is a Bayesian equilibrium, consider

$$\begin{aligned} u_i(\sigma'_i, \sigma_{-i}^*) &= \sum_{\theta_i \in \Theta_i} P(\theta_i) u_i(\sigma'_i(\theta_i), \sigma_{-i}^* | \theta_i) \\ &\leq \sum_{\theta_i \in \Theta_i} P(\theta_i) u_i(\sigma_i^*(\theta_i), \sigma_{-i}^* | \theta_i), \text{ since } (\sigma^*, P) \text{ is a BE} \\ &= u_i(\sigma_i^*, \sigma_{-i}^*) \end{aligned}$$



# Equivalence of equilibrium concepts



## Theorem

*In finite Bayesian games, a strategy profile is Bayesian Equilibrium iff it is a Nash equilibrium*

# Equivalence of equilibrium concepts



## Theorem

*In finite Bayesian games, a strategy profile is Bayesian Equilibrium iff it is a Nash equilibrium*

## Proof.

For the reverse direction, proof by contradiction. Suppose  $(\sigma^*, P)$  is not a Bayesian equilibrium i.e., there exists some  $i \in N$ , some  $\theta_i \in \Theta_i$ , some  $a_i \in A_i$ , s.t.

$$u_i(a_i, \sigma_{-i}^* | \theta_i) > u_i(\sigma_i^*(\theta_i), \sigma_{-i}^* | \theta_i)$$

# Equivalence of equilibrium concepts



## Theorem

*In finite Bayesian games, a strategy profile is Bayesian Equilibrium iff it is a Nash equilibrium*

## Proof.

For the reverse direction, proof by contradiction. Suppose  $(\sigma^*, P)$  is not a Bayesian equilibrium i.e., there exists some  $i \in N$ , some  $\theta_i \in \Theta_i$ , some  $a_i \in A_i$ , s.t.

$$u_i(a_i, \sigma_{-i}^* | \theta_i) > u_i(\sigma_i^*(\theta_i), \sigma_{-i}^* | \theta_i)$$

Construct the strategy  $\hat{\sigma}_i$  s.t.,

$$\hat{\sigma}_i(\theta'_i) = \sigma_i^*(\theta'_i), \forall \theta'_i \in \Theta_i \setminus \{\theta_i\}$$



# Equivalence of equilibrium concepts

## Theorem

*In finite Bayesian games, a strategy profile is Bayesian Equilibrium iff it is a Nash equilibrium*

## Proof.

For the reverse direction, proof by contradiction. Suppose  $(\sigma^*, P)$  is not a Bayesian equilibrium i.e., there exists some  $i \in N$ , some  $\theta_i \in \Theta_i$ , some  $a_i \in A_i$ , s.t.

$$u_i(a_i, \sigma_{-i}^* | \theta_i) > u_i(\sigma_i^*(\theta_i), \sigma_{-i}^* | \theta_i)$$

Construct the strategy  $\hat{\sigma}_i$  s.t.,

$$\hat{\sigma}_i(\theta'_i) = \sigma_i^*(\theta'_i), \forall \theta'_i \in \Theta_i \setminus \{\theta_i\}$$

$$\hat{\sigma}_i(\theta_i)[a_i] = 1, \hat{\sigma}_i(\theta_i)[b_i] = 0, \forall b_i \in A_i \setminus \{a_i\}$$



# Equivalence of equilibrium concepts



## Theorem

*In finite Bayesian games, a strategy profile is Bayesian Equilibrium iff it is a Nash equilibrium*

## Proof.

Reverse direction proof continued ...



# Equivalence of equilibrium concepts



## Theorem

*In finite Bayesian games, a strategy profile is Bayesian Equilibrium iff it is a Nash equilibrium*

## Proof.

Reverse direction proof continued ...

$$u_i(\hat{\sigma}_i, \sigma_{-i}^*) = \sum_{\tilde{\theta}_i \in \Theta_i} P(\tilde{\theta}_i) u_i(\hat{\sigma}_i(\tilde{\theta}_i), \sigma_{-i}^* | \tilde{\theta}_i)$$



# Equivalence of equilibrium concepts



## Theorem

*In finite Bayesian games, a strategy profile is Bayesian Equilibrium iff it is a Nash equilibrium*

## Proof.

Reverse direction proof continued ...

$$\begin{aligned} u_i(\hat{\sigma}_i, \sigma_{-i}^*) &= \sum_{\tilde{\theta}_i \in \Theta_i} P(\tilde{\theta}_i) u_i(\hat{\sigma}_i(\tilde{\theta}_i), \sigma_{-i}^* | \tilde{\theta}_i) \\ &= \sum_{\tilde{\theta}_i \in \Theta_i \setminus \{\theta_i\}} P(\tilde{\theta}_i) u_i(\hat{\sigma}_i(\tilde{\theta}_i), \sigma_{-i}^* | \tilde{\theta}_i) \end{aligned}$$



# Equivalence of equilibrium concepts



## Theorem

*In finite Bayesian games, a strategy profile is Bayesian Equilibrium iff it is a Nash equilibrium*

## Proof.

Reverse direction proof continued ...

$$\begin{aligned} u_i(\hat{\sigma}_i, \sigma_{-i}^*) &= \sum_{\tilde{\theta}_i \in \Theta_i} P(\tilde{\theta}_i) u_i(\hat{\sigma}_i(\tilde{\theta}_i), \sigma_{-i}^* | \tilde{\theta}_i) \\ &= \sum_{\tilde{\theta}_i \in \Theta_i \setminus \{\theta_i\}} P(\tilde{\theta}_i) u_i(\hat{\sigma}_i(\tilde{\theta}_i), \sigma_{-i}^* | \tilde{\theta}_i) + P(\theta_i) u_i(\hat{\sigma}_i(\theta_i), \sigma_{-i}^* | \theta_i) \end{aligned}$$



# Equivalence of equilibrium concepts



## Theorem

*In finite Bayesian games, a strategy profile is Bayesian Equilibrium iff it is a Nash equilibrium*

## Proof.

Reverse direction proof continued ...

$$\begin{aligned} u_i(\hat{\sigma}_i, \sigma_{-i}^*) &= \sum_{\tilde{\theta}_i \in \Theta_i} P(\tilde{\theta}_i) u_i(\hat{\sigma}_i(\tilde{\theta}_i), \sigma_{-i}^* | \tilde{\theta}_i) \\ &= \sum_{\tilde{\theta}_i \in \Theta_i \setminus \{\theta_i\}} P(\tilde{\theta}_i) u_i(\hat{\sigma}_i(\tilde{\theta}_i), \sigma_{-i}^* | \tilde{\theta}_i) + P(\theta_i) u_i(\hat{\sigma}_i(\theta_i), \sigma_{-i}^* | \theta_i) \\ &> \sum_{\tilde{\theta}_i \in \Theta_i \setminus \{\theta_i\}} P(\tilde{\theta}_i) u_i(\sigma_i^*(\tilde{\theta}_i), \sigma_{-i}^* | \tilde{\theta}_i) + P(\theta_i) u_i(\sigma_i^*(\theta_i), \sigma_{-i}^* | \theta_i) = u_i(\sigma_i^*, \sigma_{-i}^*) \end{aligned}$$



# Equivalence of equilibrium concepts



## Theorem

*In finite Bayesian games, a strategy profile is Bayesian Equilibrium iff it is a Nash equilibrium*

## Proof.

Reverse direction proof continued ...

$$\begin{aligned} u_i(\hat{\sigma}_i, \sigma_{-i}^*) &= \sum_{\tilde{\theta}_i \in \Theta_i} P(\tilde{\theta}_i) u_i(\hat{\sigma}_i(\tilde{\theta}_i), \sigma_{-i}^* | \tilde{\theta}_i) \\ &= \sum_{\tilde{\theta}_i \in \Theta_i \setminus \{\theta_i\}} P(\tilde{\theta}_i) u_i(\hat{\sigma}_i(\tilde{\theta}_i), \sigma_{-i}^* | \tilde{\theta}_i) + P(\theta_i) u_i(\hat{\sigma}_i(\theta_i), \sigma_{-i}^* | \theta_i) \\ &> \sum_{\tilde{\theta}_i \in \Theta_i \setminus \{\theta_i\}} P(\tilde{\theta}_i) u_i(\sigma_i^*(\tilde{\theta}_i), \sigma_{-i}^* | \tilde{\theta}_i) + P(\theta_i) u_i(\sigma_i^*(\theta_i), \sigma_{-i}^* | \theta_i) = u_i(\sigma_i^*, \sigma_{-i}^*) \end{aligned}$$

Hence,  $(\sigma_i^*, \sigma_{-i}^*)$  is not a Nash equilibrium

□

# Existence of Bayesian Equilibrium



## Theorem

*Every finite Bayesian game has a Bayesian equilibrium.*

[Finite Bayesian game: set of players, action set and type set are finite]

# Existence of Bayesian Equilibrium



## Theorem

*Every finite Bayesian game has a Bayesian equilibrium.*

[Finite Bayesian game: set of players, action set and type set are finite]

## Proof.

Proof idea: Transform the Bayesian game into a complete information game treating each type as a player, and invoke Nash Theorem for the existence of equilibrium - which is a BE in the original game. [See addendum for details] □



- ▶ Equilibrium in IIEFGs
- ▶ Game Theory in Practice: P2P File Sharing
- ▶ Bayesian Games
- ▶ Strategy, Utility in Bayesian Games
- ▶ Equilibrium in Bayesian Games
- ▶ **Examples in Bayesian Equilibrium**

## Example 2 : Sealed Bid Auction



Two players, both willing to buy an object. Their values and bids lie in  $[0, 1]$ .

### Allocation Function

$$O_1(b_1, b_2) = \mathbb{1}\{b_1 \geq b_2\}$$

$$O_2(b_1, b_2) = \mathbb{1}\{b_2 > b_1\}$$

### Beliefs

$$f(\theta_2|\theta_1) = 1, \forall \theta_1, \theta_2$$

$$f(\theta_1|\theta_2) = 1, \forall \theta_1, \theta_2$$

$$f(\theta_1, \theta_2) = 1, \forall \theta_1, \theta_2$$



- If  $b_1 \geq b_2$ , player 1 wins and pays her bid; otherwise, player 2 wins and pays her bid.

$$u_1(b_1, b_2, \theta_1, \theta_2) = (\theta_1 - b_1)\mathbb{1}\{b_1 \geq b_2\}$$

$$u_2(b_1, b_2, \theta_1, \theta_2) = (\theta_2 - b_2)\mathbb{1}\{b_1 < b_2\}$$



- If  $b_1 \geq b_2$ , player 1 wins and pays her bid; otherwise, player 2 wins and pays her bid.

$$u_1(b_1, b_2, \theta_1, \theta_2) = (\theta_1 - b_1)\mathbb{1}\{b_1 \geq b_2\}$$

$$u_2(b_1, b_2, \theta_1, \theta_2) = (\theta_2 - b_2)\mathbb{1}\{b_1 < b_2\}$$

- $b_1 = s_1(\theta_1), b_2 = s_2(\theta_2)$

Assume  $s_i(\theta_i) = \alpha_i \theta_i, \alpha_i > 0, i = 1, 2$

# First Price Auction



To find the BE, we need to find the  $s_i^*$  (or  $\alpha_i^*$ ) that maximizes the ex-interim utility of player  $i$ . i.e.

$$\max_{\sigma_i} u_i(\sigma_i, \sigma_{-i}^* | \theta_i)$$

For player 1, this reduces to

$$\begin{aligned} \max_{\sigma_i} u_i(\sigma_i, \sigma_{-i}^* | \theta_i) &= \max_{b_1 \in [0,1]} \int_0^1 f(\theta_2 | \theta_1) (\theta_1 - b_1) \mathbb{1}\{b_1 \geq \alpha_2 \theta_2\} d\theta_2 \\ &= \max_{b_1 \in [0,1]} (\theta_1 - b_1) \frac{b_1}{\alpha_2} \\ \implies b_1 &= \frac{\theta_1}{2} \end{aligned}$$

# First Price Auction



From this we get,

$$s_1^*(\theta_1) = \frac{\theta_1}{2}$$
$$s_2^*(\theta_2) = \frac{\theta_2}{2}$$

is a BE.

In the Bayesian Game induced by uniform prior on first price auction, bidding half the true value is a Bayesian equilibrium.

# Second Price Auction



Highest bidder wins but pays the second highest bid.

$$u_1(b_1, b_2, \theta_1, \theta_2) = (\theta_1 - b_2)\mathbb{1}\{b_1 \geq b_2\}$$

$$u_2(b_1, b_2, \theta_1, \theta_2) = (\theta_2 - b_1)\mathbb{1}\{b_1 < b_2\}$$

# Second Price Auction



Highest bidder wins but pays the second highest bid.

$$u_1(b_1, b_2, \theta_1, \theta_2) = (\theta_1 - b_2)\mathbb{1}\{b_1 \geq b_2\}$$

$$u_2(b_1, b_2, \theta_1, \theta_2) = (\theta_2 - b_1)\mathbb{1}\{b_1 < b_2\}$$

Player 1 has to maximize

$$= \int_0^1 f(\theta_2|\theta_1)(\theta_1 - s_2(\theta_2))\mathbb{1}\{b_1 \geq s_2(\theta_2)\}d\theta_2$$

$$= \int_0^1 1 \cdot (\theta_1 - \alpha_2\theta_2)\mathbb{1}\{\theta_2 \leq \frac{b_1}{\alpha_2}\}d\theta_2$$

$$= \frac{1}{\alpha_2}(b_1\theta_1 - \frac{\theta_1^2}{2})$$

This is maximized when  $b_1 = \theta_1$ . Similarly for  $b_2 = \theta_2$ .

# Second Price Auction



If the distribution of  $\theta_1$  and  $\theta_2$  were arbitrary but independent, the maximization problem would have been

$$\int_0^{\frac{b_1}{\alpha_2}} f(\theta_2)(\theta_1 - \alpha_2\theta_2)d\theta_2 = \theta_1 F\left(\frac{b_1}{\alpha_2}\right) - \alpha_2 \int_0^{\frac{b_1}{\alpha_2}} \theta_2 f(\theta_2)d\theta_2$$

# Second Price Auction



If the distribution of  $\theta_1$  and  $\theta_2$  were arbitrary but independent, the maximization problem would have been

$$\int_0^{\frac{b_1}{\alpha_2}} f(\theta_2)(\theta_1 - \alpha_2\theta_2)d\theta_2 = \theta_1 F\left(\frac{b_1}{\alpha_2}\right) - \alpha_2 \int_0^{\frac{b_1}{\alpha_2}} \theta_2 f(\theta_2)d\theta_2$$

Differentiating w.r.t.  $b_1$ , we get

$$\theta_1 \frac{1}{\alpha_2} f\left(\frac{b_1}{\alpha_2}\right) - \alpha_2 \cdot \frac{b_1}{\alpha_2} f\left(\frac{b_1}{\alpha_2}\right) \frac{1}{\alpha_2} = 0 \implies f\left(\frac{b_1}{\alpha_2}\right) (b_1 - \theta_1) = 0 \quad (1)$$

$$\implies b_1 = \theta_1, \text{ if } f\left(\frac{b_1}{\alpha_2}\right) > 0 \quad (2)$$

Similarly for player 2.

# Second Price Auction



If the distribution of  $\theta_1$  and  $\theta_2$  were arbitrary but independent, the maximization problem would have been

$$\int_0^{\frac{b_1}{\alpha_2}} f(\theta_2)(\theta_1 - \alpha_2\theta_2)d\theta_2 = \theta_1 F\left(\frac{b_1}{\alpha_2}\right) - \alpha_2 \int_0^{\frac{b_1}{\alpha_2}} \theta_2 f(\theta_2)d\theta_2$$

Differentiating w.r.t.  $b_1$ , we get

$$\theta_1 \frac{1}{\alpha_2} f\left(\frac{b_1}{\alpha_2}\right) - \alpha_2 \cdot \frac{b_1}{\alpha_2} f\left(\frac{b_1}{\alpha_2}\right) \frac{1}{\alpha_2} = 0 \implies f\left(\frac{b_1}{\alpha_2}\right) (b_1 - \theta_1) = 0 \quad (1)$$

$$\implies b_1 = \theta_1, \text{ if } f\left(\frac{b_1}{\alpha_2}\right) > 0 \quad (2)$$

Similarly for player 2.

**For any independent positive prior, bidding true type is a BE of the induced Bayesian game in Second Price Auction.**



भारतीय प्रौद्योगिकी संस्थान मुंबई  
**Indian Institute of Technology Bombay**