भारतीय प्रौद्योगिकी संस्थान मुंबई

Indian Institute of Technology Bombay

CS 6001: Game Theory and Algorithmic Mechanism Design

Week 7

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Slide preparation acknowledgments: C. R. Pradhit and Adit Akarsh

ज्ञानम् परमम् ध्येयम्

Knowledge is the supreme goal

Contents

Mechanism Design (Inverse Game Theory)

The objective/desired are set - the task is to set the rules of the game. Examples : election, licensing scarce resource (spectrum, cloud), matching students to universities.

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- θ_i : value of i for the object.
- $u_i(x, \theta_i) = a_i\theta_i - p_i$

Social Choice Function

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Answer: We need mechanisms, but they can be complicated

Mechanisms

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An **indirect** mechanism is a collection of message spaces and a decision rule $\langle M_1, M_2, \ldots, M_n, g \rangle$

- M_i is the message space of agent i
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Answer

Due to a result that will follow.

Weakly Dominant

Definition

In a mechanism $\langle M_1, M_2, \ldots, M_n, g \rangle$, a message m_i is **weakly dominant** for player i at θ_i if

$$u_i(g(m_i, \tilde{m}_{-i}), \theta_i) \geqslant u_i(g(m_i', \tilde{m}_{-i}), \theta_i), \forall \tilde{m}_{-i}, \forall m_i'$$

All subsequent definitions assume cardinal preferences, however they can be replaced with ordinal, e.g., the above one could be defined as

$$g(m_i, \tilde{m}_{-i}) \; \theta_i \; g(m_i', \tilde{m}_{-i}), \forall \tilde{m}_{-i}, \forall m_i'$$

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An SCF $f : \Theta \to X$ is **implemented in dominant strategies** by $\langle M_1, M_2, \ldots, M_n, g \rangle$ if

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An SCF $f : \Theta \rightarrow X$ is **implemented in dominant strategies** by $\langle M_1, M_2, \ldots, M_n, g \rangle$ if

- $\exists$ message mappings $s_i : \Theta_i \rightarrow M_i$, s.t, $s_i(\theta_i)$ is a dominant strategy for agent i at θ_i, $\forall \theta_i \in \Theta_i$, $\forall i \in N$.

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We call this an indirect implementation, i.e., SCF f is **dominant strategy implementable (DSI)** by $\langle M_1, M_2, \ldots, M_n, g \rangle$.

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A direct mechanism $\langle \Theta_1, \Theta_2, \ldots, \Theta_n, f \rangle$ is **dominant strategy incentive compatible (DSIC)** if

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- But luckily, there is a result that reduces the search space.

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$\Rightarrow f$ is DSIC. □

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- Recall : Bayesian games $\langle N, (M_i)_{i \in N}, (\Theta_i)_{i \in N}, P, (\Gamma_\theta)_{\theta \in \Theta} \rangle$

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Observation: If an SCF f dominant strategy implementable, then it is Bayesian implementable.

Bayesian Incentive Compatible

Definition

A direct mechanism $\langle \Theta_1, \Theta_2, \ldots, \Theta_n, f \rangle$ is **Bayesian Incentive Compatible (BIC)** if

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Revelation Principle for Bayesian Implementable SCFs

Revelation Principle (for Bayesian implementable SCFs)

If an SCF f is implementable in Bayesian equilibrium, then f is BIC.

- Proof idea is similar to the DSI, with expected utilities at appropriate places.
- For truthfulness of these two kinds, we will only consider incentive compatibility.
- These results hold even for ordinal preferences and mechanisms.
- Detailed proof: homework

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Arrow's Social Welfare Function Setup

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- Finite set of alternatives $A = \{a_1, a_2, \ldots, a_m\}$
- Finite set of players $N = \{1, 2, \ldots, n\}$
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  1. **asymmetric** part P_i
  2. **symmetric** part I_i
- Example:

$$R_i = \begin{bmatrix} a \\ b,c \\ d \end{bmatrix} = \{(a,b),(a,c),(a,d),(b,c),(c,b),(b,d),(c,d)\}$$

$$\Rightarrow P_i = \begin{bmatrix} a & a \\ b & c \\ d & d \end{bmatrix} = \{(a,b),(a,c),(a,d),(b,d),(c,d)\}, \qquad I_i = \{(b,c),(c,b)\}$$

Arrovian Social Welfare Function (ASWF)

$$F : \mathcal{R}^n \to \mathcal{R} \qquad \text{domain and co-domain are both rankings}$$

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Pareto or Unanimity

Definition (Weak Pareto)

An ASWF F satisfies **weak Pareto** if the following holds for all $a, b \in A$ and for every preference profile R:

$$[aP_i b,\ \forall i \in N] \implies [a\hat{F}(R)b].$$

(Read as: whenever the LHS (the ‘if’ part) holds, the RHS holds as well)

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Which property implies the other?

Independence of Irrelevant Alternatives

- We say $R_i, R'_i \in \mathcal{R}$ **agree** on $\{a, b\}$ for agent i if

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If the relative positions of two alternatives are the same in two different preference profiles, then the aggregate should also match the relative positions of those two alternatives

Example

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R

a	a	c	d
b	c	b	c
c	b	a	b
d	d	d	a

R'

d	c	b	b
a	a	c	a
b	b	a	d
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R					R'			
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b	c	b	c		a	a	c	a
c	b	a	b		b	b	a	d
d	d	d	a		c	d	d	c

- IIA says $F(R)|_{a,b} = F(R')|_{a,b}$
- Simple aggregation rules, e.g., **scoring rules**: each position of each agent gets a score $(s_1, s_2, \ldots, s_m), s_i \geqslant s_{i+1}, i = 1, 2, \ldots, m-1$, the final ordering is in the decreasing order of the scores

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- One special scoring rule: **plurality**, $s_1 = 1, s_i = 0, i = 2, \ldots, m$.

Satisfaction of IIA

Question

Does plurality satisfy IIA?

R			
a	a	c	d
b	c	b	c
c	b	a	b
d	d	d	a

R′			
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Question

Does **dictatorship** satisfy IIA?

A **dictatorship** ASWF is where there exists a pre-determined agent d and $F^d(R) = R_d$

Arrow's impossibility result

Theorem (Arrow 1951)

For $|A| \geqslant 3$, if an ASWF F satisfies WP and IIA, then it must be dictatorial.

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We cannot aggregate reasonably even when there is no truthfulness constraint

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Observation: $D_G(a, b) \implies \overline{D}_G(a, b)$

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Note: these two lemmas immediately prove the theorem

Field expansion lemma

Lemma

Let F satisfy WP and IIA, then $\forall a, b, x, y, G \subseteq N, G \neq \emptyset, a \neq b, x \neq y$

$$\overline{D}_G(a, b) \Rightarrow D_G(x, y).$$

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It implies that under WP and IIA, the two notions of decisiveness are equivalent.

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Proof of FEL

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- Construct R' s.t.

G		$N \setminus G$	
a	a	b	b
$\vdots$	$\vdots$	$\vdots$	$\vdots$
b	b	a	y
$\vdots$	$\vdots$	$\vdots$	$\vdots$
y	y	y	a

positions of a and y in $N \setminus G$ s.t. $R'|_{a,y} = R|_{a,y}$

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$\vdots$	$\vdots$	$\vdots$	$\vdots$
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$$
\begin{array}{cc||cc}
\multicolumn{2}{c}{G} & \multicolumn{2}{c}{N \setminus G} \\
\hline\hline
a & a & b & b \\
\vdots & \vdots & \vdots & \vdots \\
b & b & a & y \\
\vdots & \vdots & \vdots & \vdots \\
y & y & y & a
\end{array}
$$

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- IIA $\Rightarrow a\hat{F}(R)y$. Hence, $D_G(a,y)$

Proof of FEL (contd.)

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G		$N \setminus G$	
x	x	x	b
$\vdots$	$\vdots$	$\vdots$	$\vdots$
a	a	b	x
$\vdots$	$\vdots$	$\vdots$	$\vdots$
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x	x	x	b
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a	a	b	x
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$\vdots$	$\vdots$	$\vdots$	$\vdots$
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- Transitivity $\Rightarrow x\hat{F}(R')b$
- IIA $\Rightarrow x\hat{F}(R)b$. Hence, $D_G(x,b)$

Proof of FEL (other cases)

- **Case 3:** $\overline{D}_G(a,b) \overset{\text{(case 1)}}{\Longrightarrow} D_G(a,y)\ (y \neq a,b) \overset{\text{(definition)}}{\Longrightarrow} \overline{D}_G(a,y) \overset{\text{(case 2)}}{\Longrightarrow} D_G(x,y)\ (x \neq a,y)$
- **Case 4:** $\overline{D}_G(a,b) \overset{\text{(case 2)}}{\Longrightarrow} D_G(x,b)\ (x \neq a,b) \overset{\text{(definition)}}{\Longrightarrow} \overline{D}_G(x,b) \overset{\text{(case 1)}}{\Longrightarrow} D_G(x,a)\ (x \neq a,b)$
- **Case 5:** $\overline{D}_G(a,b) \overset{\text{(case 1)}}{\Longrightarrow} D_G(a,y)\ (y \neq a,b) \overset{\text{(definition)}}{\Longrightarrow} \overline{D}_G(a,y) \overset{\text{(case 2)}}{\Longrightarrow} D_G(b,y)\ (y \neq a,b)$
- **Case 6:** $\overline{D}_G(a,b) \overset{\text{(case 2)}}{\Longrightarrow} D_G(x,b)\ (x \neq a,b) \overset{\text{(definition)}}{\Longrightarrow} \overline{D}_G(x,b) \overset{\text{(case 2)}}{\Longrightarrow} D_G(a,b)$
- **Case 7:** $\overline{D}_G(a,b) \overset{\text{(case 5)}}{\Longrightarrow} D_G(b,y)\ (y \neq a,b) \overset{\text{(definition)}}{\Longrightarrow} \overline{D}_G(b,y) \overset{\text{(case 1)}}{\Longrightarrow} D_G(b,a)$

Group contraction lemma

Lemma

Let F satisfy WP and IIA, and let $G \subseteq N, G \neq \emptyset, |G| \geqslant 2$ *be decisive. Then* $\exists G' \subset G, G' \neq \emptyset$ *which is also decisive.*

Proof:

- $G, |G| \geqslant 2$ is given. Let $G_1 \subset G, G_2 = G \setminus G_1$, $G_1, G_2 \neq \emptyset$, arbitrary.

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- Construct R

G_1	G_2	$N \setminus G$
a	c	b
b	a	c
c	b	a

$aP_ib, \forall i \in G$ and G decisive $\Rightarrow a\hat{F}(R)b$

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$$aP_ib, \forall i \in G \text{ and } G \text{ decisive } \Rightarrow a\hat{F}(R)b$$

- Where can c stand in $F(R)$ w.r.t. a? We will show in every possible case, either G_1 or G_2 will be decisive

Proof of GCL

Case 1: $a\hat{F}(R)c$

G_1	G_2	$N \setminus G$
a	c	b
b	a	c
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have seen $\Rightarrow a\hat{F}(R)b$

- Consider G_1

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- Consider each R' where the above relation holds

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- by IIA $a\hat{F}(R')c$
- Hence $\overline{D}_{G_1}(a,c) \overset{\text{(FEL)}}{\Longrightarrow} D_{G_1}$

Proof of GCL (contd.)

Case 2: $\neg(a\hat{F}(R)c) \implies cF(R)a$

- $a\hat{F}(R)b$ and $cF(R)a$ give $c\hat{F}(R)b$

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G_1	G_2	$N \setminus G$
a	c	b
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- $a\hat{F}(R)b$ and $cF(R)a$ give $c\hat{F}(R)b$
- Consider G_2

Proof of GCL (contd.)

Case 2: $\neg(a\hat{F}(R)c) \implies cF(R)a$

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- Consider G_2
- $cP_ib, \forall i \in G_2, bP_ic, \forall i \in N \setminus G_2$

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- $cP_ib, \forall i \in G_2, bP_ic, \forall i \in N \setminus G_2$
- Consider each R' where the above relation holds
- by IIA $c\hat{F}(R')b$
- Hence $\overline{D}_{G_2}(c,b) \overset{\text{(FEL)}}{\implies} D_{G_2}$

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