भारतीय प्रौद्योगिकी संस्थान मुंबई

Indian Institute of Technology Bombay

CS 6001: Game Theory and Algorithmic Mechanism Design

Week 8

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Slide preparation acknowledgments: C. R. Pradhit and Adit Akarsh

ज्ञानम् परमम् ध्येयम्

Knowledge is the supreme goal

Contents

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- Ways out:
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  2. put restrictions on agent preferences
- **Social choice function** (SCF)

$$f : \mathcal{P}^n \to A$$

$A = \{a_1, a_2, \ldots, a_m\}$ Finite set of alternatives

$N = \{1, 2, \ldots, n\}$ Finite set of players

$\mathcal{P}$ Set of all **linear** preference ordering

Examples

- Most representative: **voting**

$$\begin{array}{cccc} \multicolumn{4}{c}{P} \\ \hline\hline a & a & c & d \\ b & b & b & c \\ c & c & d & b \\ d & d & a & a \end{array} \quad \xrightarrow{f} \quad A = \{a, b, c, d\}$$

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  - ***k*-approval**: $(\underbrace{1, 1, \ldots, 1}_{k}, 0, 0, \ldots, 0)$

Examples (contd.)

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- **Copeland**: based on Copeland score = number of wins in pairwise elections

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no **scoring rule** is Condorcet consistent

Desirable properties of SCF

- Recall, **social choice function**, $f : \mathcal{P}^n \to A$
- **Pareto domination**: an alternative a is **Pareto dominated** by b if $\forall i \in N, bP_i a$ (also, a is called Pareto dominated if some such b exists)

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An SCF f is *Pareto efficient* (PE) if $\forall P$ and $a \in A$, if a is Pareto dominated, then $f(P) \neq a$.

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An SCF f is *unanimous* (UN) if $\forall P$ satisfying $P_1(1) = P_2(1) = \ldots = P_n(1) = a$ [$P_i(k)$ is the k-th favorite alternative of i], it holds that $f(P) = a$.

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Which implies which? if the top choice of all voters is the same, say a, all other alternatives are Pareto dominated by a

Desirable properties of SCF (contd.)

Definition (Onto)

An SCF f is *onto* (ONTO) if $\forall a \in A$, $\exists P^{(a)} \in \mathcal{P}^n$ s.t. $f(P^{(a)}) = a$.

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- Plurality with fixed tie-breaking
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- Copeland with fixed tie-breaking $a \succ b \succ c$, Copeland score = number of wins in pairwise elections

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Strategyproofness and its implications

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$$D(a, P_i) := \{b \in A : aP_ib\}$$

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- The set of alternatives *below* a in P_i

$$P_i = \begin{matrix} b \\ a \\ c \\ d \end{matrix} \quad \Rightarrow \quad D(a, P_i) = \{c, d\}$$

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Definition (Monotonicity)

An SCF is *monotone* (MONO) if for every two profiles P and P' that satisfy $f(P) = a$ and $D(a, P_i) \subseteq D(a, P'_i)$, for all $i \in N$, it holds that $f(P') = a$.

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*An SCF f is **strategyproof** iff it is **monotone**.*

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- P and P' with $f(P) = a$ and $D(a, P_i) \subseteq D(a, P_i') \ \forall i \in N$

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$$\begin{array}{ccccc}
(P_1, P_2, P_3, \dots, P_n) & \rightarrow & (P'_1, P_2, P_3, \dots P_n) & \rightarrow & (P'_1, P'_2, P_3, \dots, P_n) \\
P = P^{(0)} & & P^{(1)} & & P^{(2)} \\
\cdots & \rightarrow & (P'_1, \dots P'_k, P_{k+1}, \dots P_n) & \rightarrow & (P'_1 \cdots P'_n) \\
 & & P^{(k)} & & P^{(n)} = P'
\end{array}$$

Proof of SP $\Leftrightarrow$ MONO

$$\begin{array}{ccccc}
(P_1, P_2, P_3, \ldots, P_n) & \rightarrow & (P'_1, P_2, P_3, \ldots P_n) & \rightarrow & (P'_1, P'_2, P_3, \ldots, P_n) \\
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Proof of SP $\Leftrightarrow$ MONO

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(P_1, P_2, P_3, \ldots, P_n) & \rightarrow & (P'_1, P_2, P_3, \ldots P_n) & \rightarrow & (P'_1, P'_2, P_3, \ldots, P_n) \\
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$$\begin{array}{ccccc}
\underset{P = P^{(0)}}{(P_1, P_2, P_3, \ldots, P_n)} & \rightarrow & \underset{P^{(1)}}{(P'_1, P_2, P_3, \ldots P_n)} & \rightarrow & \underset{P^{(2)}}{(P'_1, P'_2, P_3, \ldots, P_n)} \\
\cdots & \rightarrow & \underset{P^{(k)}}{(P'_1, \ldots P'_k, P_{k+1}, \ldots P_n)} & \rightarrow & \underset{P^{(n)} = P'}{(P'_1 \cdots P'_n)}
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  3. $b \ P_k \ a$ and $a \ P'_k \ b \rightarrow$ voter k misreports in both

Proof of SP $\Leftrightarrow$ MONO

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  1. $a \ P_k \ b$ and $a \ P'_k \ b \rightarrow$ voter k misreports $P'_k \rightarrow P_k$
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  3. $b \ P_k \ a$ and $a \ P'_k \ b \rightarrow$ voter k misreports in both
- Contradiction to f being SP

Proof of SP $\Leftrightarrow$ MONO (contd.)

- For (SP $\impliedby$ MONO), we will prove $\neg$SP $\implies$ $\neg$MONO

Proof of SP $\Leftrightarrow$ MONO (contd.)

- For (SP $\Longleftarrow$ MONO), we will prove $\neg$SP $\Longrightarrow$ $\neg$MONO
- Suppose not, i.e., f is $\neg$SP but MONO

Proof of SP $\Leftrightarrow$ MONO (contd.)

- For (SP $\impliedby$ MONO), we will prove ¬SP $\implies$ ¬MONO
- Suppose not, i.e., f is ¬SP but MONO
- ¬SP implies that $\exists i, P_i, P_i', P_{-i}$, s.t. $\underbrace{f(P_i', P_{-i})}_{b \text{ (say)}} \; P_i \; \underbrace{f(P_i, P_{-i})}_{a \text{ (say)}} = b \; P_i \; a$

Proof of SP $\Leftrightarrow$ MONO (contd.)

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- Construct P'' s.t. $P''_{-i} = P_{-i}, P_i''(1) = b$, $P_i''(2) = a$
- Consider two transitions:
  1. $(P_i, P_{-i}) \rightarrow (P_i'', P_{-i})$

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- Construct P'' s.t. $P''_{-i} = P_{-i}, P_i''(1) = b$, $P_i''(2) = a$
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  2. $(P_i', P_{-i}) \rightarrow (P_i'', P_{-i})$
     $D(b, P_i') \subseteq D(b, P_i'') \xrightarrow{\text{MONO}} f(P_i'', P_{-i}) = b$ (contradiction)

Proof of SP $\Leftrightarrow$ MONO (contd.)

- For (SP $\Longleftarrow$ MONO), we will prove $\neg$SP $\Longrightarrow$ $\neg$MONO
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- Construct P'' s.t. $P''_{-i} = P_{-i}, P''_i(1) = b, P''_i(2) = a$
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  2. $(P'_i, P_{-i}) \to (P''_i, P_{-i})$
     $D(b, P'_i) \subseteq D(b, P''_i) \xRightarrow{\text{MONO}} f(P''_i, P_{-i}) = b$ (contradiction)
- This concludes the proof

Equivalence of PE, UN, ONTO under SP

Lemma

If an SCF f is MONO and ONTO, then f is PE.

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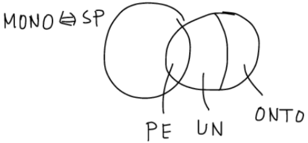

Figure: Relation between SCFs

Proof

- Suppose not, i.e. f is MONO and ONTO but not PE then $\exists\ a,\ b,\ P$ s.t., $b\ P_i\ a\ \forall i \in N$ but $f(P) = a$

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- Clearly, $D(b, P'_i) \subseteq D(b, P''_i)\ \forall i \in N \xRightarrow{MONO} f(P'') = b$
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- Construct P'' s.t. $P_i''(1) = b$, $P_i''(2) = a$, $\forall i \in N$
- Since f is ONTO, $\exists P'$, s.t., $f(P') = b$
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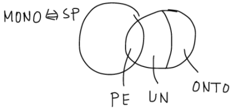
Corollary: f is SP+PE $\iff$ f is SP+UN $\iff$ f is SP+ONTO



Contents

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- No reasonable voting rule is **truthful**
- Plurality, Borda, Copeland, Maximin, ...
- **Crucial**: the preferences are **unrestricted**, i.e., all $m!$ preference profiles are in the domain of the SCF f

Contents

- The Social Choice Setup
- The Gibbard-Satterthwaite Theorem
- Proof of Gibbard-Satterthwaite Theorem
- Domain Restriction
- Median Voting Rule
- Median Voter Theorem: Part 1
- Median Voter Theorem: Part 2

Points to note

1. $|A| = 2$: GS theorem does not hold. Plurality with a fixed tie breaking rule is SP, ONTO, and non-dictatorial

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Points to note

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3. **Indifference in preferences**: in general, GS theorem does not hold. In the proof, we use some specific constructions. If they are possible, then GS theorem holds.
4. **Cardinalization**: GS theorem will hold as long as all possible ordinal ranks are feasible in the cardinal preferences.

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Proof:

- If $P_1(1) = P_2(1)$, then UN implies $f(P) = P_1(1)$ (ONTO $\iff$ UN under SP)
- Say $P_1(1) = a \neq b = P_2(1)$. For contradiction assume $f(P) = c \neq a, b$ (need at least 3 alternatives)

Proof of GS Theorem (contd.)

P_1	P_2	P_1	P'_2	P'_1	P'_2	P'_1	P_2
a	b	a	b	a	b	a	b
.	.	.	a	b	a	b	.
.	.	.	.	.	.	.	.

$$f(P_1, P_2) = c(\neq a, b)$$

- Now $f(P_1, P'_2) \in \{a, b\}$ [because all alternatives except b are Pareto dominated by a]

Proof of GS Theorem (contd.)

P_1	P_2	P_1	P_2'	P_1'	P_2'	P_1'	P_2
a	b	a	b	a	b	a	b
.	.	.	a	b	a	b	.
.	.	.	.	.	.	.	.

$$f(P_1, P_2) = c(\neq a, b)$$

- Now $f(P_1, P_2') \in \{a, b\}$ [because all alternatives except b are Pareto dominated by a]
- But if $f(P_1, P_2') = b$, then player 2 manipulates from P_2 to P_2', hence $f(P_1 \; P_2') = a$

Proof of GS Theorem (contd.)

P_1	P_2	P_1	P_2'	P_1'	P_2'	P_1'	P_2
a	b	a	b	a	b	a	b
.	.	.	a	b	a	b	.
.	.	.	.	.	.	.	.

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- By a similar argument, $f(P_1', P_2) = b$

Proof of GS Theorem (contd.)

P_1	P_2	P_1	P'_2	P'_1	P'_2	P'_1	P_2
a	b	a	b	a	b	a	b
.	.	.	a	b	a	b	.
.	.	.	.	.	.	.	.

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Proof of GS Theorem (contd.)

P_1	P_2	P_1	P_2'	P_1'	P_2'	P_1'	P_2
a	b	a	b	a	b	a	b
.	.	.	a	b	a	b	.
.	.	.	.	.	.	.	.

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    - $P_1', P_2 \rightarrow P_1', P_2'$ outcome should be b

Proof of GS Theorem (contd.)

P_1	P_2	P_1	P_2'	P_1'	P_2'	P_1'	P_2
a	b	a	b	a	b	a	b
.	.	.	a	b	a	b	.
.	.	.	.	.	.	.	.

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- Now $f(P_1, P_2') \in \{a, b\}$ [because all alternatives except b are Pareto dominated by a]
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- By a similar argument, $f(P_1', P_2) = b$
- Now apply MONO
  - $P_1', P_2 \rightarrow P_1', P_2'$ outcome should be b
  - $P_1, P_2' \rightarrow P_1', P_2'$ outcome should be a (contradiction)

Proof of GS Theorem (contd.)

Lemma (Two player version of GS theorem)

Suppose $|A| \geqslant 3$, $N = \{1, 2\}$, *and* f *is ONTO and SP*

- *Let* $P : P_1(1) = a \neq b = P_2(1)$, $P' : P'(1) = c$, $P'_2(1) = d$
- *If* $f(P) = a$, *then* $f(P') = c$
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Proof of GS Theorem (contd.)

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Proof: If $c = d$, unanimity proved the lemma. Hence consider $c \neq d$.

cases ↓	c	d
1	a	b
2	$\neq a, b$	b
3	$\neq a, b$	$\neq b$
4	a	$\neq a, b$
5	b	$\neq a, b$
6	b	a

- Enough to consider the case: if $f(P) = a \implies f(P') = c$
- The other case is symmetric
- These cases are exhaustive

Proof of GS Theorem (contd.)

Case 1: $c = a$, $d = b$,

P_1	P_2	P'_1	P'_2	$\hat{P}_1$	$\hat{P}_2$
a	b	a	b	a	b
.	.	.	.	b	a
.	.	.	.	.	.

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$$\underset{a}{P_1\ P_2} \xrightarrow{MONO} \underset{a}{\hat{P}_1\ \hat{P}_2} \qquad\qquad \underset{b}{P'_1\ P'_2} \xrightarrow{MONO} \underset{b}{\hat{P}_1\ \hat{P}_2}$$

Proof of GS Theorem (contd.)

Case 2: $c \neq a, b, d = b$,

P_1	P_2	P'_1	P'_2	$\hat{P}_1$	P_2
a	b	c	b	c	b
.	.	.	.	a	.
.	.	.	.	.	.

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(apply case 1)

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$$\begin{array}{c} P_1' \; P_2' \\ b \end{array} \xrightarrow{MONO} \begin{array}{c} \hat{P}_1 \; P_2 \\ b \end{array}$$

(apply case 1)

agent 1 misreports $\hat{P}_1 \to P_1$ as $a \; \hat{P}_1 \; b$

Proof of GS Theorem (contd.)

Case 3: $c \neq a, b$, and $d \neq b$,

P_1	P_2	P'_1	P'_2	$\hat{P}_1$	$\hat{P}_2$
a	b	c	d	c	b
.	.	.	.	.	.
.	.	.	.	.	.

- Say $f(P') = d$

$P' \rightarrow \hat{P}$ $\quad$ $f(\hat{P}) = b$ (case 2)

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Proof of GS Theorem (contd.)

Case 4: $c = a$, and $d \neq b, a$

P_1	P_2	P'_1	P'_2	$\hat{P}_1$	$\hat{P}_2$
a	b	$c = a$	d	a	b
.	.	.	.	.	.
.	.	.	.	.	.

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$$P \rightarrow \hat{P} \qquad f(\hat{P}) = a \text{ (case 1)}$$

Proof of GS Theorem (contd.)

Case 5: $c = b$, and $d \neq b, a$

P_1	P_2	P'_1	P'_2	$\hat{P}_1$	$\hat{P}_2$	P_1	$\hat{P}_2$
a	b	$c = b$	d	c	d	a	d
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.

- Say $f(P') = d$

$$P' \to \hat{P} \qquad f(\hat{P}) = d \text{ (case 1)}$$

$$P \to (P_1, \hat{P}_2) \qquad f(P_1, \hat{P}_2) = a \text{ (case 4)}$$

$$(P_1, \hat{P}_2) \to \hat{P} \qquad f(\hat{P}) = a \text{ (case 2)}, b \neq a, d$$

Proof of GS Theorem (contd.)

Case 6: $c = b$, and $d = a$ (this case proof acknowledgments: Tanish Agarwal)

P_1	P_2	P'_1	P'_2	$\hat{P}_1$	$\hat{P}_2$
a	b	b	a	c	b
.	.	.	.	.	.
.	.	.	.	.	.

where $c \neq a, b$; assume for contradiction, $f(P') = a$

Consider the transitions

$$\begin{aligned} P &\to \hat{P}, & f(\hat{P}) &= c \text{ (case 2)} \\ P' &\to \hat{P}, & f(\hat{P}) &= b \text{ (case 5)} \end{aligned}$$

leads to a contradiction. $n \geqslant 3$ **agent case**: induction on the number of agents. See Sen (2001): “A direct proof of GS theorem”, Economics Letters

Contents

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- If we reduce the set of feasible preferences from $\mathcal{P}$ to $\mathcal{S} \subset \mathcal{P}$
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  - but there can potentially be more f's that can be strategyproof on the **restricted domain**

Domain restrictions

1. Single peaked preferences
2. Divisible goods allocation
3. Quasi-linear preferences

Each of these domains have interesting non-dictatorial SCFs that are strategyproof

Single peaked preferences

- Temperature of a room

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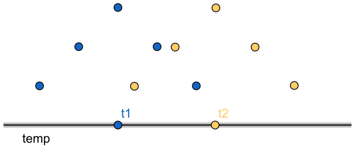

Figure: Single peaked temperature preference

Single peaked preferences

- One **common order** over the alternatives

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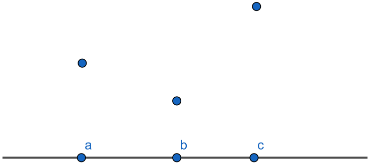
  2. consider only one-dimensional single-peakedness

Single peaked preferences

How is it a domain restriction?



Consider $a < b < c$, all possible orderings:

$$\begin{array}{cccccc} a & b & b & c & a & c \\ b & a & c & b & c & a \\ c & c & a & a & b & b \end{array}$$

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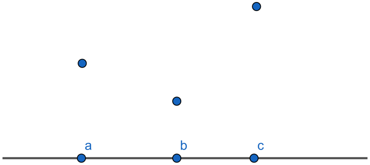

Consider $a < b < c$, all possible orderings:

a	b	b	c	a	c
b	a	c	b	c	a
c	c	a	a	b	b

Definition (Single peaked preferences)

A preference ordering P_i (linear over A) of agent i is single-peaked w.r.t. the common order $<$ of the alternatives if

1. $\forall b, c \in A$ with $b < c \leqslant P_i(1)$, cP_ib
2. $\forall b, c \in A$ with $P_i(1) \leqslant b < c$, bP_ic

Single peaked preferences

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Answer

Each player's preference has a peak. Suppose, f picks the leftmost peak. For the agent having the leftmost peak, no reason to misreport. For any other agent, the only way she can change the outcome is by reporting her peak to be left of the leftmost – but that is strictly worse than the current outcome.

Repeat this argument for any fixed k^{th} peak from left. Even the rightmost peak choosing SCF is also strategyproof, so is the median ($k = \left[\frac{n}{2}\right]$)

Contents

Median voter SCF

Definition

An SCF $f : \mathcal{S}^n \to A$ is a median voter SCF if there exists $B = \{y_1, y_2, \ldots, y_{n-1}\}$ s.t. $f(P) =$ median$(B,$ peaks$(P))$ for all preference profiles $P \in \mathcal{S}^n$.

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- Phantom voters give a complete spectrum of the median voter SCFs

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Theorem (Moulin 1980)

Every median voter SCF is strategyproof.

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Proof Sketch:

- if $f(P) = a$ and a player has a peak $P_i(1)$ to the left of a, it has no benefit by misreporting the peak to be on the right of a, which is the only way of changing the outcome of f
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Note: mean does not have this property

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Claim

Let p_{min} *and* p_{max} *be the leftmost and rightmost peaks of* P *according to* $<$*, then* f *is PE iff* $f(P) \in [p_{\text{min}}, p_{\text{max}}]$

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($\impliedby$) If $f(P) \in [p_{\min}, p_{\max}]$, then the condition $bP_i f(P), \forall i \in N$ never occurs – there does not exist an alternative b that Pareto dominates $f(P)$. Hence $f(P)$ is PE (vacuously true from definition).

Median voter SCF and Monotonicity

Definition (Monotonicity)

An SCF is *monotone* (MONO) if for every two profiles P and P' that satisfy $f(P) = a$ and $D(a, P_i) \subseteq D(a, P'_i)$, for all $i \in N$, it holds that $f(P') = a$.

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P			
a	a	c	d
b	b	b	c
c	c	d	b
d	d	a	a

P'			
c	a	c	d
b	b	b	c
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d	d	a	a

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This proof is similar to the previous one. To prove the reverse implication one needs to argue why the construction is valid in the single peaked domain. **(or provide counterexample)**

Equivalence of ONTO, UN, and PE

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Let $f : \mathcal{S}^n \to A$ is a SP SCF. Then, f is ONTO $\iff$ f is UN $\iff$ f is PE

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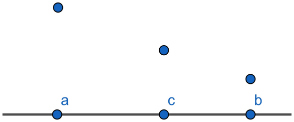

Figure: Arrangement of a, b, c

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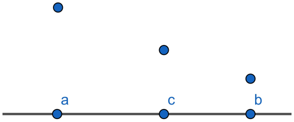

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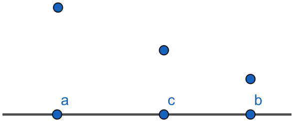

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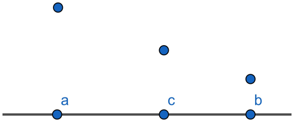

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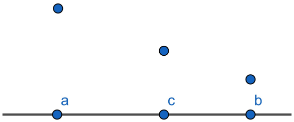
- Since preferences are single peaked, $\exists$ another alternative $c \in A$, which is a neighbour of b s.t. $c\ P_i\ b\ \forall i \in N$ (c can be a itself)



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Now, we are interested in non-dictatorial SCFs, hence a necessary property is **anonymity**

Anonymity

- **Anonymity**: outcome insensitive to agent identities

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- **Example:** $N = \{1, 2, 3\}, \sigma : \sigma(1) = 2,\ \sigma(2) = 3,\ \sigma(3) = 1$

P_1	P_2	P_3	P_1^σ	P_2^σ	P_3^σ
a	b	b	b	a	b
b	a	c	c	b	a
c	c	a	a	c	c

Anonymity (contd.)

Definition

An SCF $f : \mathcal{S}^n \to A$ is **anonymous** (ANON) if for every profile P and for **every** permutation of the agents $\sigma, f(P^\sigma) = f(P)$

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Dictatorship is not anonymous

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Proof: ($\Longleftarrow$)

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Proof: ($\Longleftarrow$)

- Median voter SCF is SP (previous theorem)
- It is **anonymous**: if we permute the agents with peaks unchanged, the outcome does not change
- It is ONTO, pick any arbitrary alternative a, put peaks of all players at a: the outcome will be a irrespective of the positions of the phantom peaks (since there are $(n-1)$ phantom peaks and n agent peaks)

Proof (contd.)

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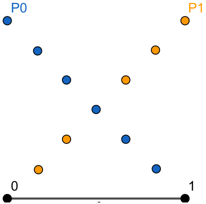

Figure: Two preferences

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- First construct phantom peaks

$$y_j = f(\underbrace{P_1^0, P_2^0, \ldots, P_{n-j}^0}_{n-j \text{ peaks leftmost}}, \underbrace{P_{n-j+1}^1, \ldots, P_n^1}_{j \text{ peaks rightmost}}),\ j = 1, \ldots, n-1$$

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$\longleftarrow$ spectrum of the peaks $\longrightarrow$

$(j-1)$ phantom	y_j	$(n-1-j)$ phantom
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- Hence, $p_1 \leqslant \cdots \leqslant p_{n-j} \leqslant y_j = a \leqslant p_{n-j+1} \leqslant \cdots \leqslant p_n$

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- repeat this argument for the first $(n-j)$ agents to get

$$f(P_1, P_2, \ldots, P_{n-j}, P_{n-j+1}^1, \ldots, P_n^1) = y_j$$

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- Hence,

$$f(P_1, \ldots, P_n) = y_j$$

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- **Claim:** Suppose f satisfies SP, ONTO, ANON, then $f(P) = \text{median}(p_1, \ldots, p_n, y_1, \ldots, y_{n-1})$
- WLOG, can assume $p_1 \leqslant p_2 \leqslant \cdots \leqslant p_n$ due to ANON
- Consider $a = \text{median}(p_1, \ldots, p_n, y_1, \ldots, y_{n-1})$
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- Since peaks of P_1 and P'_1 are the same, if x, y are on the same side of the peak, they must be the same, as the domain is single peaked
- The only other possibility is that x and y fall on different sides of the peak: **we show that this is not possible.**

Proof (contd.)

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Profile: $(P_1, P_2) = P, P_1(1) = a, P_2(1) = b$, y_1 is the phantom peak, and by assumption, median(a, b, y_1) is an agent peak

- WLOG assume that the median is a

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- By PE, c must be within a and b
- We have two cases to consider: $b < a < y_1$ and $y_1 < a < b$

Proof (contd.)

Case 2.1: $b < a < y_1$, by PE $c < a$

- Construct P_1' s.t. $P_1'(1) = a = P_1(1)$ and $y_1 \; P_1' \; c$ (possible since they are on different sides of a)

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- Agent 1 manipulates $P_1' \to P_1^1$, contradiction to f being SP

Proof (contd.)

Case 2.2: $y_1 < a < b$, by PE $a < c$

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Case 2.2: $y_1 < a < b$, by PE $a < c$

- Construct P_1' s.t. $P_1'(1) = a = P_1(1)$ and $y_1 \; P_1' \; c$
- $f(P_1', P_2) = c$ (by claim)

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- This completes the proof for two agents (case 2)
- For the generalization to n players, see Moulin (1980), "On strategyproofness and single-peakedness"

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