भारतीय प्रौद्योगिकी संस्थान मुंबई

Indian Institute of Technology Bombay

CS 6001: Game Theory and Algorithmic Mechanism Design

Week 9

Swaprava Nath

Slide preparation acknowledgments: Rounak Dalmia

ज्ञानम् परमम् ध्येयम्

Knowledge is the supreme goal

Contents

- Task Allocation Domain
- The Uniform Rule
- Mechanism Design with Transfers
- Quasi Linear Preferences
- Pareto Optimality and Groves Payments

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  - Net payoff = $w t_i - c_i t_i^2 \implies$ **maximized** at $t_i = w/2c_i$,
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- There cannot be a single common order over the alternatives s.t. the preferences are single-peaked for all agents

Task Allocation Domain and Pareto Efficiency

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Definition (Pareto Efficiency)

An SCF f is *Pareto efficient* (PE) if there does not exist any profile P where there exists a task allocation $a \in A$ such that it is weakly preferred over $f(P)$ by all agents and strictly preferred by at least one. Mathematically,

$$\nexists a \in A \text{ s.t. } \begin{array}{ll} a \, R_i \, f(P) & \forall i \in N, \\ a \, P_j \, f(P) & \exists j \in N \end{array}$$

Implications of Pareto Efficiency

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Answer

No. If such a j exists, increasing k's share of task and reducing j's makes both players strictly better off
Therefore, $\forall j \in N, f_j(P) \leqslant p_j$

3. If $\sum_{i \in N} p_i < 1$, by a similar argument, we conclude that $\forall j \in N, f_j(P) \geqslant p_j$

Task Allocation Domain and Anonymity

Definition (Anonymity)

An SCF f is *anonymous* (ANON) if for every agent permutation $\sum_{i \in N} : N \to N$, the task shares get permuted accordingly, i.e.,

$$\forall \sigma,\ f_{\sigma(j)}(P^{\sigma}) = f_j(P)$$

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Example:

- $N = \{1, 2, 3\},\ \sigma(1) = 2, \sigma(2) = 3, \sigma(3) = 1$
- $P = (0.7, 0.4, 0.3) \implies P^{\sigma} = (0.3, 0.7, 0.4)$
- $f_1(0.7, 0.4, 0.3) = f_2(0.3, 0.7, 0.4)$

Task Allocation Domain: Some Candidate SCFs

Definition (Serial Dictatorship)

A predetermined sequence of the agents is fixed. Each agent is given either her peak share or the leftover share of the task. If $\sum_{i \in N} p_i < 1$, then the last agent is given the leftover share.

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Not ANON. Also quite unfair to the last agent.

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if the report is $0.1, 0.3, 0.1$, $c = 1/0.5$, player 1 gets 0.2

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  - $f_i^u(P) \leqslant p_i, \forall i \in N$, if $\sum_{i \in N} p_i > 1$
- This is PE from our previous observation on PE: *allocations should stay on the same side of the peaks for every agent*

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- See **Sprumont (1991) : Division problem with single-peaked preferences**

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  4. Partitioning indivisible objects, S = set of objects, $A = \{(A_1, \cdots, A_n) : A_i \subseteq S,\ \forall i \in N, A_i \cap A_j = \emptyset,\ \forall i \neq j\}$

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  - if type changes to 'business' θ_i^{bus}, $v_i(B, \theta_i^{\text{bus}}) > v_i(P, \theta_i^{\text{bus}})$

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- **Payments** $\pi_i \in \mathbb{R}, \ \forall i \in N$
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- Utility of player i, when its type is θ_i, and the outcome is $x = (a, \pi)$ is given by

$$u_i((a, \pi), \theta_i) = v_i(a, \theta_i) - \pi_i \qquad \textbf{(quasi-linear payoff)}$$

Quasi Linear Domain

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- This restriction opens up possibilities of several non-dictatorial mechanisms

Contents

- Task Allocation Domain
- The Uniform Rule
- Mechanism Design with Transfers
- Quasi Linear Preferences
- Pareto Optimality and Groves Payments

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- **Payment function**

$$p_i : \Theta_1 \times \Theta_2 \times \cdots \Theta_n \to \mathbb{R}, \forall i \in N$$

When the types are $\theta_i, i \in N, p_i(\theta_1, \cdots, \theta_n) = \pi_i \in \mathbb{R}$

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5. **Max-min/egalitarian**

$$f^{MM}(\theta) \in \arg\max_{a \in A} \min_{i \in N} v_i(a, \theta_i)$$

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Definition (DSIC)

A mechanism (f, p) is **dominant strategy incentive compatible (DSIC)** if

$$v_i(f(\theta_i, \tilde{\theta}_{-i}), \theta_i) - p_i(\theta_i, \tilde{\theta}_{-i}) \geqslant v_i(f(\theta_i', \tilde{\theta}_{-i}), \theta_i) - p_i(\theta_i', \tilde{\theta}_{-i}), \forall \tilde{\theta}_{-i} \in \Theta_{-i}, \theta_i', \theta_i \in \Theta_i, \forall i \in N$$

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- If we can find a payment that implements an allocation rule, there exists uncountably many payments that can implement it
- **The converse question:** when do the payments that implement f differ only by a factor $h_i(\theta_{-i})$?

Properties of the Payment

- Suppose the allocation is same in two type profiles θ and $\tilde{\theta} = (\tilde{\theta}_i, \theta_{-i})$
- i.e., $f(\theta) = f(\tilde{\theta}) = a$, then
- if p implements f, then $p_i(\theta) = p_i(\tilde{\theta})$ **[exercise]**

Contents

- Task Allocation Domain
- The Uniform Rule
- Mechanism Design with Transfers
- Quasi Linear Preferences
- Pareto Optimality and Groves Payments

Pareto Optimality in Quasi-linear domain

Definition (Pareto Optimal)

A mechanism $(f, (p_1, \dots, p_n))$ is **Pareto optimal** if at any type profile $\theta \in \Theta$, there does not exist an allocation $b \neq f(\theta)$ and payments $(\pi_1, \dots, \pi_n)$ with $\sum_{i \in N} \pi_i \geqslant \sum_{i \in N} p_i(\theta)$ s.t.,

$$v_i(b, \theta_i) - \pi_i \geqslant v_i(f(\theta), \theta_i) - p_i(\theta), \forall i \in N,$$

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- Pareto optimality is meaningless if there is no restriction on the payment
- One can always put excessive subsidy to every agent to make everyone better off
- So, the condition requires to spend at least the same budget

Pareto Optimality in Quasi-linear Domain

Theorem

*A mechanism $(f,(p_1,\cdots,p_n))$ is **Pareto optimal** iff it is allocatively efficient*

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- summing over the all these inequalities

$$\sum_{i\in N} v_i(b,\theta_i)-\sum_{i\in N}\pi_i > \sum_{i\in N} v_i(f(\theta),\theta_i)-\sum_{i\in N} p_i(\theta)$$

$$\sum_{i\in N} v_i(b,\theta_i)-\sum_{i\in N} v_i(f(\theta),\theta_i) > \sum_{i\in N}\pi_i-\sum_{i\in N} p_i(\theta)\geqslant 0$$

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- f is $\neg$AE

Proof (contd.)

- $(\implies)$ $\neg$AE $\implies$ $\neg$PO

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- also $\sum_{i \in N} \pi_i = \sum_{i \in N} p_i(\theta)$
- Hence f is not PO

Allocatively Efficient Rule is Implementable

- Consider the following payment: $p_i^G(\theta_i, \theta_{-i}) = h_i(\theta_{-i}) - \sum_{j \neq i} v_j(f^{AE}(\theta_i, \theta_{-i}), \theta_j)$, where $h_i : \Theta_{-i} \to \mathbb{R}$ is an arbitrary function: **Groves payment**

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Example

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- $\theta_1 = 10$, $\theta_2 = 8$, $\theta_3 = 6$, $\theta_4 = 4$, when they get the object, zero otherwise

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- $p_1 = 4 - 0 = 4, p_2 = 4 - 10 = -6, p_3 = 4 - 10 = -6, p_4 = 6 - 10 = -4$, i.e., only player 1 pays, other get paid

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- **Surprisingly, this is a truthful mechanism**

Groves mechanisms are Truthful

Theorem

Groves mechanisms are DSIC

- Consider player i

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- Consider player i
- $f^{AE}(\theta_i, \tilde{\theta}_{-i}) = a$, and $f^{AE}(\theta_i', \tilde{\theta}_{-i}) = b$

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- utility of player i when he reports θ_i is

$$
\begin{aligned}
& v_i(f^{AE}(\theta_i, \tilde{\theta}_{-i}), \theta_i) - p_i(\theta_i, \tilde{\theta}_{-i}) \\
& = v_i(f^{AE}(\theta_i, \tilde{\theta}_{-i}), \theta_i) - h_i(\tilde{\theta}_{-i}) + \sum_{j \neq i} v_j(f^{AE}(\theta_i, \tilde{\theta}_{-i}), \tilde{\theta}_j) \\
& \geqslant v_i(f^{AE}(\theta_i', \tilde{\theta}_{-i}), \theta_i) - h_i(\tilde{\theta}_{-i}) + \sum_{j \neq i} v_j(f^{AE}(\theta_i', \tilde{\theta}_{-i}), \tilde{\theta}_j) \\
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& \geqslant v_i(f^{AE}(\theta'_i, \tilde{\theta}_{-i}), \theta_i) - h_i(\tilde{\theta}_{-i}) + \sum_{j \neq i} v_j(f^{AE}(\theta'_i, \tilde{\theta}_{-i}), \tilde{\theta}_j) \\
& = v_i(f^{AE}(\theta'_i, \tilde{\theta}_{-i}), \theta_i) - p_i(\theta'_i, \tilde{\theta}_{-i})
\end{aligned}
$$

- Since player i was arbitrary, this holds for all $i \in N$. Hence the claim.

भारतीय प्रौद्योगिकी संस्थान मुंबई

Indian Institute of Technology Bombay