भारतीय प्रौद्योगिकी संस्थान मुंबई

Indian Institute of Technology Bombay

CS 6001: Game Theory and Algorithmic Mechanism Design

Week 9

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Slide preparation acknowledgments: Rounak Dalmia

ज्ञानम् परमम् ध्येयम्

Knowledge is the supreme goal

Contents

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  - Net payoff = $wt_i - c_i t_i^2 \implies$ **maximized** at $t_i = w/2c_i$, and **monotone** decreasing on both sides

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- There cannot be a single common order over the alternatives s.t. the preferences are single-peaked for all agents

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Definition (Pareto Efficiency)

An SCF f is *Pareto efficient* (PE) if there does not exist any profile P where there exists a task allocation $a \in A$ such that it is weakly preferred over $f(P)$ by all agents and strictly preferred by at least one. Mathematically,

$$\nexists P, \text{ where } \exists a \in A \text{ s.t. } \begin{array}{ll} a \, R_i \, f(P) & \forall i \in N, \\ a \, P_j \, f(P) & \exists j \in N. \end{array}$$

Implications of Pareto Efficiency

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Question

Can there be an agent j s.t. $f_j(P) > p_j$ if f is PE?

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Question

Can there be an agent j s.t. $f_j(P) > p_j$ if f is PE?

Answer

No. If such a j exists, increasing k's share of task and reducing j's makes both players strictly better off
Therefore, $\forall j \in N, f_j(P) \leqslant p_j$

3. If $\sum_{i \in N} p_i < 1$, by a similar argument, we conclude that $\forall j \in N, f_j(P) \geqslant p_j$

Task Allocation Domain and Anonymity

Definition (Anonymity)

An SCF f is *anonymous* (ANON) if for every agent permutation $\sigma : N \to N$, the task shares get permuted accordingly, i.e.,

$$\forall \sigma, \; f_{\sigma(j)}(P^{\sigma}) = f_j(P), \forall j \in N.$$

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Example:

- $N = \{1, 2, 3\},\ \sigma(1) = 2, \sigma(2) = 3, \sigma(3) = 1$
- $P = (0.7, 0.4, 0.3) \implies P^{\sigma} = (0.3, 0.7, 0.4)$

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- $f_1(0.7, 0.4, 0.3) = f_2(0.3, 0.7, 0.4)$
- $f_2(0.7, 0.4, 0.3) = f_3(0.3, 0.7, 0.4)$
- $f_3(0.7, 0.4, 0.3) = f_1(0.3, 0.7, 0.4)$

Strategyproofness

Manipulability: an SCF f is **manipulable** if $\exists i \in N$ and a profile P such that, $f(P_i', P_{-i})\ P_i\ f(P_i, P_{-i})$, for some P_i'.

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Strategyproofness (equivalent definition):

$$f(P_i, P_{-i}) \; P_i \; f(P'_i, P_{-i}) \quad \text{OR} \quad f_i(P_i, P_{-i}) = f_i(P'_i, P_{-i}), \forall P_i, P'_i \in T, \forall i \in N, \forall P_{-i} \in T^{n-1}.$$

Task Allocation Domain: Some Candidate SCFs

Definition (Serial Dictatorship)

A predetermined sequence of the agents is fixed. Each agent is given either her peak share or the leftover share of the task. If $\sum_{i \in N} p_i < 1$, then the last agent is given the leftover share.

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Answer

Not ANON. Also quite unfair to the last agent.

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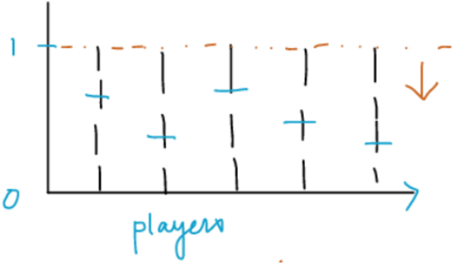

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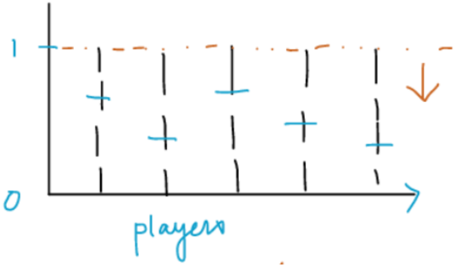

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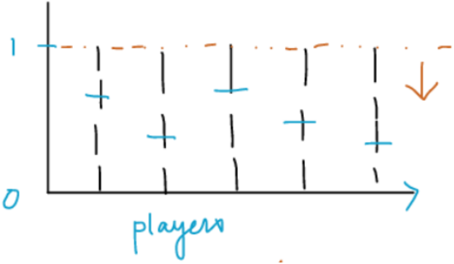

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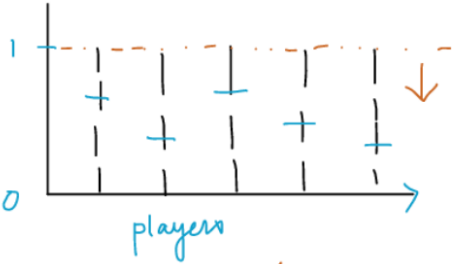

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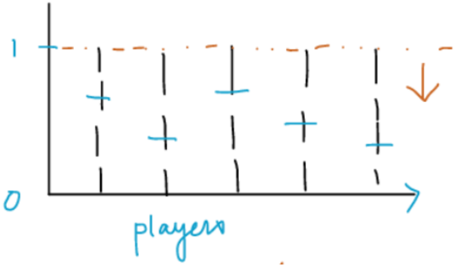
- Symmetric for $\sum_{i \in N} p_i > 1$



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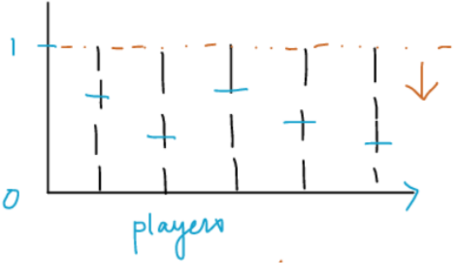

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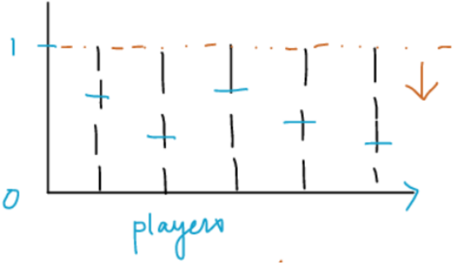

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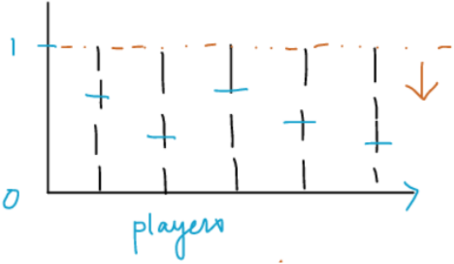

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  - $f_i^u(P) \leqslant p_i, \forall i \in N$, if $\sum_{i \in N} p_i > 1$
- This is PE from our previous observation on PE: *allocations should stay on the same side of the peaks for every agent*

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- Manipulation, only for $i \in N$ s.t. $f_i^u(P) > p_i \implies \mu(P) > p_i$
- The only way i can change the allocation is by reporting $p_i' > \mu(P) > p_i$

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- The Uniform Rule
- Mechanism Design with Transfers
- Quasi Linear Preferences
- Pareto Optimality and Groves Payments

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  4. Partitioning indivisible objects, S = set of objects, $A = \{(A_1, \cdots, A_n) : A_i \subseteq S,\ \forall i \in N, A_i \cap A_j = \emptyset,\ \forall i \neq j\}$

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  - if type changes to 'business' θ_i^{bus}, $v_i(B, \theta_i^{\text{bus}}) > v_i(P, \theta_i^{\text{bus}})$

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- Utility of player i, when its type is θ_i, and the outcome is $x = (a, \pi)$ is given by

$$u_i((a, \pi), \theta_i) = v_i(a, \theta_i) - \pi_i \qquad \textbf{(quasi-linear payoff)}$$

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Definition (DSIC)

A mechanism (f, p) is **dominant strategy incentive compatible (DSIC)** if

$$v_i(f(\theta_i, \tilde{\theta}_{-i}), \theta_i) - p_i(\theta_i, \tilde{\theta}_{-i}) \geqslant v_i(f(\theta_i', \tilde{\theta}_{-i}), \theta_i) - p_i(\theta_i', \tilde{\theta}_{-i}), \forall \tilde{\theta}_{-i} \in \Theta_{-i}, \theta_i', \theta_i \in \Theta_i, \forall i \in N$$

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$$v_1(f(\theta^H, \theta_2), \theta^H) - p_1(\theta^H, \theta_2) \geqslant v_1(f(\theta^L, \theta_2), \theta^H) - p_1(\theta^L, \theta_2), \forall \theta_2 \in \Theta_2 \quad (1)$$

$$v_1(f(\theta^L, \theta_2), \theta^L) - p_1(\theta^L, \theta_2) \geqslant v_1(f(\theta^H, \theta_2), \theta^L) - p_1(\theta^H, \theta_2), \forall \theta_2 \in \Theta_2 \quad (2)$$

Implications of DSIC

Question

What needs to be satisfied for a DSIC mechanism (f, p)?

Example

$N = \{1,2\}, \Theta_1 = \Theta_2 = \{\theta^H, \theta^L\}, f : \Theta_1 \times \Theta_2 \to A$. The following conditions must hold

Player 1:

$$v_1(f(\theta^H, \theta_2), \theta^H) - p_1(\theta^H, \theta_2) \geqslant v_1(f(\theta^L, \theta_2), \theta^H) - p_1(\theta^L, \theta_2), \forall \theta_2 \in \Theta_2 \tag{1}$$

$$v_1(f(\theta^L, \theta_2), \theta^L) - p_1(\theta^L, \theta_2) \geqslant v_1(f(\theta^H, \theta_2), \theta^L) - p_1(\theta^H, \theta_2), \forall \theta_2 \in \Theta_2 \tag{2}$$

Player 2:

$$v_2(f(\theta^H, \theta_1), \theta^H) - p_2(\theta^H, \theta_1) \geqslant v_2(f(\theta^L, \theta_1), \theta^H) - p_2(\theta^L, \theta_1), \forall \theta_1 \in \Theta_1 \tag{3}$$

$$v_2(f(\theta^L, \theta_1), \theta^L) - p_2(\theta^L, \theta_1) \geqslant v_2(f(\theta^H, \theta_1), \theta^L) - p_2(\theta^H, \theta_1), \forall \theta_1 \in \Theta_1 \tag{4}$$

Properties of the Payment

- Say (f, p) is incentive compatible, i.e., p implements f

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- If we can find a payment that implements an allocation rule, there exists uncountably many payments that can implement it

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- If we can find a payment that implements an allocation rule, there exists uncountably many payments that can implement it
- **The converse question:** when do the payments that implement f differ only by a factor $h_i(\theta_{-i})$?

Properties of the Payment

- Suppose the allocation is same in two type profiles θ and $\tilde{\theta} = (\tilde{\theta}_i, \theta_{-i})$
- i.e., $f(\theta) = f(\tilde{\theta}) = a$, then
- if p implements f, then $p_i(\theta) = p_i(\tilde{\theta})$ **[exercise]**

Contents

- Task Allocation Domain
- The Uniform Rule
- Mechanism Design with Transfers
- Quasi Linear Preferences
- Pareto Optimality and Groves Payments

Pareto Optimality in Quasi-linear domain

Definition (Pareto Optimal)

A mechanism $(f, (p_1, \dots, p_n))$ is **Pareto optimal** if for every type profile $\theta \in \Theta$, there does not exist an allocation $b \neq f(\theta)$ and payments $(\pi_1, \dots, \pi_n)$ s.t.,

$$v_i(b, \theta_i) - \pi_i \geqslant v_i(f(\theta), \theta_i) - p_i(\theta), \forall i \in N,$$

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with the inequality being strict for some $i \in N$

- Pareto optimality is meaningless if there is no restriction on the payment
- One can always put excessive subsidy to every agent to make everyone better off
- So, the condition requires to spend at least the same budget

Pareto Optimality in Quasi-linear Domain

Theorem

A mechanism $(f, (p_1, \cdots, p_n))$ *is* ***Pareto optimal*** *iff it is allocatively efficient*

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- summing over the all these inequalities

$$\sum_{i \in N} v_i(b, \theta_i) - \sum_{i \in N} \pi_i > \sum_{i \in N} v_i(f(\theta), \theta_i) - \sum_{i \in N} p_i(\theta)$$
$$\sum_{i \in N} v_i(b, \theta_i) - \sum_{i \in N} v_i(f(\theta), \theta_i) > \sum_{i \in N} \pi_i - \sum_{i \in N} p_i(\theta) \geqslant 0$$

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A mechanism $(f, (p_1, \cdots, p_n))$ is ***Pareto optimal*** *iff it is allocatively efficient*

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- f is $\neg$AE

Proof (contd.)

- $(\implies)$ $\neg$AE $\implies$ $\neg$PO

Proof (contd.)

- $(\implies)\ \neg\text{AE} \implies \neg\text{PO}$
- $\neg\text{AE} \implies \exists\theta, b \neq f(\theta)$ s.t. $\sum_{i\in N} v_i(b,\theta_i) > \sum_{i\in N} v_i(f(\theta),\theta_i)$

Proof (contd.)

- $(\implies)$ $\neg$AE $\implies$ $\neg$PO
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- Let $\delta = \sum_{i \in N} v_i(b, \theta_i) - \sum_{i \in N} v_i(f(\theta), \theta_i) > 0$

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- also $\sum_{i \in N} \pi_i = \sum_{i \in N} p_i(\theta)$
- Hence f is not PO

Allocatively Efficient Rule is Implementable

- Consider the following payment: $p_i^G(\theta_i, \theta_{-i}) = h_i(\theta_{-i}) - \sum_{j \neq i} v_j(f^{AE}(\theta_i, \theta_{-i}), \theta_j)$, where $h_i : \Theta_{-i} \to \mathbb{R}$ is an arbitrary function: **Groves payment**

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- **Surprisingly, this is a truthful mechanism**

Groves mechanisms are Truthful

Theorem

Groves mechanisms are DSIC

- Consider player i

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- $f^{AE}(\theta_i, \tilde{\theta}_{-i}) = a$, and $f^{AE}(\theta_i', \tilde{\theta}_{-i}) = b$

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- utility of player i when he reports θ_i is

$$
\begin{aligned}
& v_i(f^{AE}(\theta_i, \tilde{\theta}_{-i}), \theta_i) - p_i(\theta_i, \tilde{\theta}_{-i}) \\
& = v_i(f^{AE}(\theta_i, \tilde{\theta}_{-i}), \theta_i) - h_i(\tilde{\theta}_{-i}) + \sum_{j \neq i} v_j(f^{AE}(\theta_i, \tilde{\theta}_{-i}), \tilde{\theta}_j) \\
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& \geqslant v_i(f^{AE}(\theta'_i, \tilde{\theta}_{-i}), \theta_i) - h_i(\tilde{\theta}_{-i}) + \sum_{j \neq i} v_j(f^{AE}(\theta'_i, \tilde{\theta}_{-i}), \tilde{\theta}_j) \\
& = v_i(f^{AE}(\theta'_i, \tilde{\theta}_{-i}), \theta_i) - p_i(\theta'_i, \tilde{\theta}_{-i})
\end{aligned}
$$

- Since player i was arbitrary, this holds for all $i \in N$. Hence the claim.

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